Motives and Milnor conjecture

notes by S. Kelly:

Day I

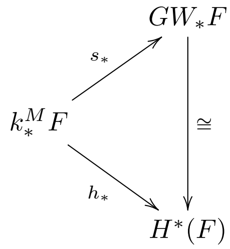
Contents

1 Galois cohomology, Milnor K-theory, and quadratic forms I

Outline : describe the three graded rings involved in the conjecture together with the maps between them used for comparison. All the material is classical and described in several books so few proofs will be given.

1.1 General introduction to the conjecture.

The origin is the paper [Mil70]. Let F be a field with $Char\ F \neq 2$. Let $k^M_*(F)$ be Milnor's K-theory mod 2 (the small k denotes mod 2), GW_*F the graded Witt ring, and $H^*(F)$ Galois cohomology with coefficients $\mathbb{Z}/2$. We have morphisms



Milnor's Conjecture. *The maps s_* and h_* are isomorphisms.*

1. Definition of Milnor's K-theory.
2. Connections with quadratic forms.
3. Functorial proportion of $k_*^n F$.
4. Galois cohomology.
5. The link between Milnor K-theory and Galois cohomology.

1.2 The graded ring $K_*^M F$

Definition. $(k^n, F, +) \cong (F^*, *)$ $\{a\} \leftrightarrow a$ subject to the relations

1. $\{ab\} = \{a\} + \{b\}$.
2. $\{1\} = 0$

Then $K_*^n F = \bigoplus_{n \geq 0} (K_1^n F)^{\otimes n} / \langle \{a\} \otimes \{1-a\}, a \in F^*, a \neq 1 \rangle$. $K_0^M \cong \mathbb{Z}$, $K_1^M F \cong F^*$.

Presentation: $K_*^M F$ is the associative ring with unit generated by $\{a\}, a \in F^*$ subject to

(DR1) $\{ab\} = \{a\} + \{b\}$.

(DR2) $\{a\} \otimes \{1-a\} = 0$.

Notation: $\{a_1\} \otimes \cdots \otimes \{a_n\} = \{a_1, \ldots, a_n\}$.

1.3 Relations and symbols

Let $a, b, a_i \in F^*$.

Lemma 1. *1.* $\{a, -a\} = 0$

2. $\{a, b\} = 0$ *if* $a + b = 0$ *or* 1

3. $\{a, b\} = \{b, a\}$

4. $\alpha\beta = (-1)^{nm}\beta\alpha$ *for* $\alpha \in K^M F, \beta \in K^M F$ *i.e.* $K_*^M F$ *is a graded commutative.*

5. $\{a_1, \dots, a_n\} = 0$ *if* $a_i + a_j = 0$ *or* 1 *for some* $i \neq j$.

6. $\{a_1, \dots, a_n\} = 0$ *if* $a_1 + \dots + a_n = 0$ *or* 1.

7. $\{a\}^2 = \{a\} \otimes \{a\} = \{a, a\} = \{a, -1\} = \{-1, a\}$.

Proof of (1). If $a = 1$ then $\{1\} = 0$. If $a \neq 1$, then $-a = \frac{1-a}{1-a^{-1}}$ so $\{-a\} = \{1-a\} - \{1-a^{-1}\}$. It follows that $\{a, -a\} = \{a, 1-a\} - \{a, 1-a^{-1}\} = \{a^{-1}, 1-a^{-1}\} = 0$. □

1.4 Milnor's K-theory mod 2

Definition. We define $k_*^M F = K_*^M F / 2K_*^M F$.

Recall that $k_0^M F \cong \mathbb{Z}/2$ and $k_1^M F = F^*/F^{*2}$ $(2\{a\} = \{a^2\}$.

Presentation: $k_*^M F$ is the associative ring with unit generated by $\{a\}, a \in F^*$ subject to

(DR1) $\{ab\} = \{a\} + \{b\}$.

(DR2) $\{a\} \otimes \{1-a\} = 0$.

(DR3) $2\{a\} = 0$.

Remarque 2.

$$\begin{aligned} \{a_1, a_2, \dots, a_i b_i^2, \dots, a_n\} &= \{a_1, a_2, \dots, a_n\} + 2\{a_1, a_2, \dots, b_i, \dots, a_n\} \\ &= \{a_1, a_2, \dots, a_n\} \in k_*^M F \end{aligned}$$

So $\{a_1, a_2, \dots, a_n\} \in k_*^M F$ for $a_i \in F^*/F^{*2}$ is well defined.

Lemma 3. $\{a, x^2 - ay^2\} = 0$ *for* $a \in F^*, x, y \in F, x^2 + ay^2 \neq 0$. *In* $k_*^M F$

$$\{a, x^2 - ay^2\} = \{a, (1 - a(\tfrac{y}{x})^2\} = \{a(\tfrac{y}{x})^2, (1 - a(\tfrac{y}{x})^2\} = 0$$

So $a \neq F^{*2}$, $b \in N_{F(\sqrt{a}/F}(F(\sqrt{a})^*)$ *then* $\{a, b\} = 0$.

1.5 The Witt ring WF

Let $\phi : V \to F$ be a (non-degenerate) quadratic form (all quadratic forms will be non-degenerate). We say that $\phi \cong \langle a_1, \dots, a_n \rangle$ if $\phi(x_1, \dots, x_n) = \sum a_i x_i^2$ in a suitable basis of V.

Recall:

1. Let $a \in F^*$. There is $v \in V$ with $\phi(v) = a$ if and only if there exist $a_2, \dots, a_n \in F^*$ such that $\phi \cong \langle a, \dots, a_n \rangle$. In this case we say ϕ represents the value a.

2. There is $v \in V$ with $v \neq 0$ such that $\phi(v) = 0$ if and only if there exist $a_3, \dots, a_n$ with $\phi \cong \langle 1, -1, a_3, \dots, a_n \rangle$. In this case we say ϕ is isotropic.

3. Let $\dim V = n = 2m$. Then there exists $W \subset V$ with $\dim W = m$ and $\phi|_W = 0$ if and only if $\phi \cong \langle 1, -1, \dots, 1, -1 \rangle \cong m\langle 1, -1 \rangle$. In this case we say that ϕ is hyperbolic.

Definition. The Witt ring of F is $W(F) = \hat{W}(F)/\{hyp.forms\}$ where $\hat{W}(F)$ is the Grothendieck group of quadratic forms with direct orthogonal sum.

We have the following properties.

1. $[\phi_1] - [\phi_2] \in WF$

2. If $\phi_1 \cong \phi_2 \oplus \phi_3$ then $[\phi_1] = [\phi_2] + [\phi_3]$ in WF

3. $[\phi] = 0$ if ϕ is hyperbolic.

We will drop the notation $[-]$.
Since quadratic forms are diagonalizable, WF is generated by $\langle a \rangle$ for $a \in F^*$.
Fact: $\otimes$ induces a product on WF which makes it a ring.

1.6 The Graded Witt ring

There is a map $\overline{dim} : WF \to \mathbb{Z}/2$ which sends ϕ to $\overline{dim\ \phi}$.

Definition. The fundamental ideal is $IF = ker\ \overline{dim}$.

We have a filtration

$$WF = I^0F \supset IF \supset I^2F \supset I^3F \supset \dots$$

and we define

Definition. $GW_*F = \bigoplus_{n \geq 0} I^nF/I^{n+1}F$.

Note that $GW_0F = WF/IF \cong \mathbb{Z}/2$ and GW_*F is a graded $ZZ/2$-algebra.

IF is additively generated by $\langle a, b \rangle \overset{WF}{=} \langle 1, -1a, b \rangle = \langle 1, b \rangle + \langle -1, a \rangle = \langle 1, b \rangle - \langle 1, -a \rangle$ where $\langle 1, -a \rangle$ represents the value 1.

Proposition 4. *1. IF is additively generated by Witt classes of $\langle 1, -a \rangle = \langle\langle a \rangle\rangle$, with $a \in F^*$.*

2. I^nF is additively generated by Witt classes of $\langle\langle a_1, \rangle\rangle \otimes \cdots \otimes \langle\langle a_n \rangle\rangle = \langle\langle a_1, \dots, a_n \rangle\rangle$ (the latter is an n-fold Pfister form and writing it in this form is called a Pfister diagonalization.

1.7 Properties of Pfister forms.

$\dim 2^n$

$$\begin{aligned}\pi &= \langle\langle a_1, \dots, a_n \rangle\rangle \\ &= \langle 1, -a_1, \dots, -a_n, a_1a_2, \dots, (-1)^n a_1 \dots, a_n \rangle \\ &= \langle -1 \rangle + \pi'\end{aligned}$$

Lemma 5. *$\pi, ainF^*$. Then there exist $a_2, \dots, a_n$ such that $\pi \cong \langle\langle a, a_1, \dots, a_n \rangle\rangle$ if and only if π' represents $-a$.*

Corollary 6. *1. If π is isotropic then it is hyperbolic.*

2. $a \in F^*$. Then π represents the value a if and only if a is a similarity factor (i.e. $\pi \cong \langle a \rangle \otimes \pi$).

3. The set of nontrivial values represented by a Pfister form is a group.

Proof. 1. $\pi = \langle 1 \rangle + \pi'$ so if π is isotropic then π' represents the value -1. This implies that $\pi = \langle\langle 1, a_2, \dots, a_n \rangle\rangle = \langle 1, -1 \rangle \otimes \langle\langle a_2, \dots, a_n \rangle\rangle$.

2. Based on the fact that $\pi_1 = \langle 1, -a \rangle =$ Norm form of $F(\sqrt{a})/F$. $\pi_1(vw) = \pi_1(v)\pi_1(w)$.

□

1.8 Milnor's K-theory and quadratic forms

Theorem 7. *There is a unique homomorphism $k_*^M F \to GW_* F$ that maps $\{a\}$ to $\langle\langle a \rangle\rangle = \langle 1, -a \rangle \in IF/I^2F$. Moreover, it is surjective.*

Proof. Uniqueness is obvious. Moreover, if it exists, $\{a_1, \dots, a_n\} \to \langle\langle a_1, \dots, a_n \rangle\rangle = \langle\langle a_1 \rangle\rangle \otimes \dots \otimes \langle\langle a_n \rangle\rangle$. $I^n F$ is additively generated by n-fold Pfister forms, so we have surjectivity.

For existence, we need to show that the relevant relations hold.

(DR2) i.e. $\{a\} \otimes \{1-a\} = 0$. So we consider $\langle\langle a, 1-a \rangle\rangle = \langle 1, -a \rangle \otimes \langle 1, a-1 \rangle = \langle 1, -a, a-1, a(1-a) \rangle$. So hyperbolic. $\langle\langle a, 1-a \rangle\rangle = 0 in WF$.

(DR1) $\{ab\} = \{a\} + \{b\}$. We are interested in $\langle\langle a \rangle\rangle + \langle\langle b \rangle\rangle - \langle\langle ab \rangle\rangle = \langle 1, -a, 1, -b, -1, ab \rangle = \langle\langle a, b \rangle\rangle \in I^2F$. So $\langle\langle a \rangle\rangle + \langle\langle b \rangle\rangle = \langle\langle ab \rangle\rangle$ in $IF/I^2F = GW_1F$.

(DR3) $2\{a\} = 0$. We have $2\langle\langle a \rangle\rangle = \langle 1, -a, 1, -a \rangle = \langle\langle a, -1 \rangle\rangle \in I^2F$.

□

2 Motivic complexes and motivic cohomology I and II

We fix k field and consider the smooth varieties over k. The idea of motivic cohomology is to construct a universal cohomology theory.

2.1 Construction of motivic cohomology

From a heuristic point of view the construction is very simple, and follows classical algebraic topology. On the category of topological spaces, to a nice topological space X we associated $\hom_{Top}(\Delta^\bullet, X) \in sSet$. We enrich the structure a bit so that we get a chain complex $\mathbb{Z}\hom_{Top}(\Delta^\bullet, X) \in sAb$ a simplicial abelian group, from which we get a chain complex and we define

$$H_n^{sing}(X, \mathbb{Z}) = H_n(\mathbb{Z}\hom_{Top}(\Delta^\bullet, X)).$$

For varieties we need a replacement for the simplicies. We take $\Delta_n^n \stackrel{def}{=} Spec\ k[T_0, \ldots, T_n]/\sum T_i = 1$. This is non-canonically isomorphic to $\mathbb{A}_k^n$ but the point is we get boundary and coboundary maps which form a simplicial object $\Delta_k^\bullet \in cosSch/k$. The problem is that there are not enough morphisms in general from Δ_k^n to a scheme X. So we go back to algebraic topology.

We use the Dold-Thom theorem which says that

$$\tilde{H}_n(X, \mathbb{Z}) \cong \pi_n(Sym^\infty(X)^+)$$

where $(-)^+$ is the group completion and $Sym^\infty(-)$ the infinite symmetric product. This is also the reduced homology groups of the simplicial abelian group $\hom_{Top}(\Delta^\bullet, Sym^\infty X)^+)$.

We define $Cor_{Top}(X, Y) = \hom_{Top}(Y, Sym^\infty X)^+ =$ the group of finite correspondences from Y to X. The formula then becomes $\tilde{H}_n(X, \mathbb{Z}) = \tilde{H}_n(Cor_{Top}(\Delta^\bullet, X))$.

Over schemes we want to define Cor.

Definition. $X, Y \in Sm/k$. We define $Cor(Y, X)$ as a subgroup of $z_{dim\ X}(X \times Y)$. It is the free abelian subgroup generated by closed irreducible subvarieties of $X \times Y$ such that the projection to X is finite, and the image dominates an irreducible component or X.

Lemma 8 (Theorem of Suslin and Voevodsky)**.** *Let p be the exponential characteristic of k. Then $Cor(Y, X) \otimes \mathbb{Z}[\frac{1}{p}] \cong Mor_{Sch}(Y, Sym^\infty X)^+ \otimes \mathbb{Z}[\frac{1}{p}]$.*

Now we can just copy the definition.

Definition. The motivic homology or Suslin homology is defined as

$$H_n^{Sus}(X) = H_n(Cor(\Delta_k^\bullet, X)$$

Exercise: Try and prove Mayer-Vietoris (use sheaf theory).

2.2 Cohomology

Back in the topological category we have $H^n(X, \mathbb{Z}) = [X, K(\mathbb{Z}, n)] = [\Sigma^{d-n} X, K(\mathbb{Z}, d)]$ for $d \geq n$. Once again, Dold-Kan tells us that $K(\mathbb{Z}, d) = Symm^\infty S^d$ where S^d is the d-dimensional sphere. Using obstruction theory, this is equal to $\pi_{d-n}(Maps(X, Symm^\infty S^d)^+)$ for $d \geq n$. This is $H_{d-n}(Cor_{Top}(X \times \Delta^\bullet, S^d))$ (by Dold-Kan).

What is S^d in algebraic geometry?

In topology, spheres naturally appear in many places, one of which is the Gysin morphism: $V \subset W$ smooth manifolds, then the Thom isomorphism is $H^n(V) \cong H^{n+d}(N_{V/W}/N_{V/W} - \{0\})$ where $N_{V/W}$ is the normal bundle and $\{0\}$ the zero section. Using excision this becomes $H^{n+d}(W/V \cap W)$.

If V is a point, what does this become? In this case $N_{V/W} = T_W$ the tangent space at that point and $T_W/T_W - 0$ which is $\mathbb{P}(TW \oplus \mathbb{R})/\mathbb{P}(TW) = \mathbb{P}^d/\mathbb{P}^{d-1} = S^d$. This we can reinterpret in the category of varieties.

Definition. We define $H^n_M(X, \mathbb{Z}(d)) = H_{2d-n}(Cor(X \times \Delta^\bullet_k, \mathbb{P}^d)/Cor(X \times \Delta^\bullet_k, \mathbb{P}^{d-1})$.

The problem is that we cannot prove anything with these definitions, not even that they are the right ones.

2.3 Complexes of presheaves with transfers

The basic objects in defining Suslin homology or motivic cohomology are complexes of presheaves with transfers. That is, contravariant functors $Sm/k^{op} \to C_\bullet Ab$ which associate to a smooth variety Y the group $Cor(Y \times \Delta^\bullet_k, X) = F(Y)$.

These complexes have transfers. In particular, for any correspondence $c \in Cor(Z, Y)$ there is an induced morphism $c^* : F(Y) \to F(X)$. So our natural world should be the derived category of the category of presheaves with transfers.

Grothendieck already tried constructing a category of motives and motivic cohomology using correspondences, but he used all of them. If a correspondence is not finite it is difficult to define a product $z_(X, Y) \times z(Y, Z) \to z(X, Z)$. We can try using the formula $\alpha, \beta \mapsto pr_{13*}pr^*_{12}\alpha \cdot pr^*_{23}\beta$ but this is only partially defined on the level of all cycles. Adequate equivalence relations provide a solution to this problem, but then new problems are created. This problem does not exist for finite correspondences.

Lemma 9.
$$Cor(X, Y) \times Cor(Y, Z) \to Cor(X, Z)$$
is well defined for all $X, Y, Z \in Sm/k$*.*

Definition. We define $SmCor(k)$ to be the category whose objects are smooth varieties and morphisms are $Cor(X, Y)$.

Note that this is an additive category with a natural tensor structure.

Definition. We define then a presheaf with transfers to be an additive presheaf of abelian groups on $SmCor(k)$.

A presheaf with transfers is just a presheaf on Sm/k on which correspondences act. We can also consider it as a functor $Sm^{op}/k \to Ab$ such that whenever we have a finite morphism $Y \xrightarrow{f} X$ there is $f_* : F(Y) \to F(X)$.

The category of presheaves with transfers is denoted PSh_{tr}. This is an abelian category and the forgetful functor $PSh_{tr} \to PSh$ is exact and faithful.

Example 10.

1. We can embed $SmCor(k) \subset PSh_{tr}$ via Yoneda $X \mapsto \mathbb{Z}_{tr}(X) = \hom_{SmCor}(-, X)$. Lemma: The $\mathbb{Z}_{tr}(X)$ are projective in PSh_{tr}.

2. Let G be an abelian group scheme over k. Then $G(X)$ is a presheaf with transfers: for any correspondence $c \in Cor(Y, X)$ gives a morphism $G(Y) \to G(X)$ where $g : Y \to G$ is sent to $c^*g : X \to Sym^n Y \to Sym^n G \to G$.

3. Higher chow groups. $CH^i(-, p)$ are presheaves with transfers.

2.4 $\mathbb{A}^1$-localization

Question: Is motivic cohomology in the way we have defined it $\mathbb{A}^1$-invariant?

Definition. $F \in PSh_{tr}$ is $\mathbb{A}^1$-invariant if $p^* : F(X) \to F(X \times \mathbb{A}^1)$ for all X. For a complex, we say that it is $\mathbb{A}^1$-local if the cohomology presheaves are $\mathbb{A}^1$-invariant.

Definition. $C^{\mathbb{A}^1} : PSh_{tr} \to sPSh_{tr}$ sends a presheaf F to $\underline{hom}(\mathbb{Z}_{tr}(\Delta^\bullet), F)$. Explicitly, $(C^{\mathbb{A}^1}F)(U) = (\ldots F(U \times \mathbb{A}^1) \rightrightarrows F(U))$.

This functor extends to complexes and we get $C^{\mathbb{A}^1} : D^-(Psh_{tr}) \to D^-(PSh_{tr})$ which maps F to $Tot\ F(\Delta^\bullet \times -)$. We then have

$$H^n_M(X, \mathbb{Z}(d)) = H_{2d-n}(C^{\mathbb{A}^1}(Cone(\mathbb{Z}_{tr}(\mathbb{P}^d) \to \mathbb{Z}_{tr}(\mathbb{P}^{d-1})[-1])(X))$$

with the correct definition of $\mathbb{Z}_{tr}(Z/W)$ this becomes

$$\hom_{D^-PSh_{tr}}(\mathbb{Z}_{tr}(X), C^{\mathbb{A}^1}(\mathbb{Z}_{tr}(\mathbb{P}^{d-1}/\mathbb{P}^d)[-2d])[n])$$

To show that $H^\bullet_M$ is $\mathbb{A}^1$-invariant it is enough to show that $C^{\mathbb{A}^1}$ applied to anything is $\mathbb{A}^1$ invariant.

Lemma 11. *For all $F \in D^-PSh_{tr}$ the object $C^{\mathbb{A}^1}(F)$ is $\mathbb{A}^1$-local.*

The lemma implies that motivic cohomology is $\mathbb{A}^1$-local.

In the abstract setting we take the thick subcategory I of D^-PSh_{tr} generated by the $Cone(\mathbb{Z}_{tr}(\mathbb{A}^1 \times X) \to \mathbb{Z}_{tr}X)$ and define D to be the orthogonal subcategory (the full subcategory of objects that have no nonzero morphisms to I). The localization functor in this setting is then just the projection $D \to I^\perp$.

Lemma 12. *If D is cocomplete, and I is generated by compact objects then this orthogonal subcategory and projection makes sense.*

We then have

$$H^n_M(X, \mathbb{Z}(d)) = \hom_{D^-PSh_{tr}/I^{\mathbb{A}^1}}(\mathbb{Z}^{\mathbb{A}^1}_{tr}(X), \mathbb{Z}^{\mathbb{A}^1}(d)[n])$$

where $\mathbb{Z}^{\mathbb{A}^1}(d)[n]$ is the image of $\mathbb{Z}_{tr}(\mathbb{P}^{d-1}/\mathbb{P}^d)[-2d]$.

Recall that we have defined

$$\begin{aligned} H^n_M(X, \mathbb{Z}(d)) &= H_{2d-n}(Cor(X \times \Delta^\bullet_k, \mathbb{P}^d)/Cor(X \times \Delta^\bullet_k, \mathbb{P}^{d-1})) \\ &= \hom_{DPSh_{tr}}(\mathbb{Z}_{tr}(X), C^{\mathbb{A}^1}(\mathbb{Z}_{tr}(\mathbb{P}^d)/\mathbb{Z}_{tr}(\mathbb{P}^{d-1})[-2d])(-n) \end{aligned}$$

where $C^{\mathbb{A}^1} : DPSh_{tr} \to \mathbb{A}^1\text{-local} = (I^{\mathbb{A}^1})^\perp$ and this is a localization so we have

$$= \hom_{DPSh_{tr}/I^{\mathbb{A}^1}}(\mathbb{Z}^{\mathbb{A}^1}_{tr}(X), \mathbb{Z}^{\mathbb{A}^1}_{tr}(\mathbb{P}^d)/\mathbb{Z}^{\mathbb{A}^1}_{tr}(\mathbb{P}^{d-1})[-2d](n))$$

2.5 Motivic localization

Challenge: Mayer-Vietoris for $H^n_M(X, \mathbb{Z}(q))$?

One way of doing this would be to write motivic cohomology as a group of morphisms in a further quotient where we take out I^{Zar} the thick subcategory generated by complexes of the form $\mathbb{Z}_{tr}(U \cap V) \to \mathbb{Z}_{tr}(U) \oplus \mathbb{Z}_{tr}(V) \to \mathbb{Z}_{tr}(U \cup V)$. We define $I^{Zar,\mathbb{A}^1}$ to be the thick subcategory generated by $I^{\mathbb{A}^1}$ and I^{Zar} and consider the quotient of $D^- PSh_{tr}$ by $I^{Zar,\mathbb{A}^1}$.

Definition.

$$DM^{eff} = D^- PSh_{tr}/I^{Zar,\mathbb{A}^1} = (I^{Zar,\mathbb{A}^1})^\perp$$

The problem is then to have an explicit description of $C^{\mathbb{A}^1,Zar} : DPSh_{tr} \to (I^{Zar,\mathbb{A}^1})^\perp$. Setting: we have $I_1, I_2 \subset D$ admissible with $I = \langle I_1, I_2 \rangle$. We say that I_1 and I_2 are compatible if

$$\begin{array}{ccccc} M_{I_2^\perp \cap I_1^\perp} & \dashrightarrow & M_{I_1^\perp} & \dashrightarrow & M_{I_2^\perp \cup I_1^\perp} \\ \uparrow & & \uparrow & & \uparrow \\ M_{I_2} & \longrightarrow & M & \longrightarrow & M_{I_2^\perp} \\ \uparrow & & \uparrow & & \uparrow \\ M_{I_2 \cap I_1} & \dashrightarrow & M_{I_1} & \dashrightarrow & M_{I_1 \cap I_2{}^\perp} \end{array}$$

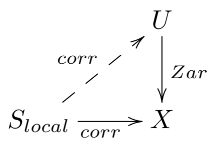
Then $C^I = C^{I_1} C^{I_2} = C^{I_2} C^{I_1}$.

If (T, τ) is a site, then sheafification $PSh \leftrightarrows Sh^\tau$ induces $DPSh/I^\tau \cong DSh^\tau$ where I^τ are the locally acyclic complexes, i.e. complexes of presheaves where the stalks are acyclic. $C^\tau : DPSh \to (I^\tau) \perp$ is the Godement resolution.

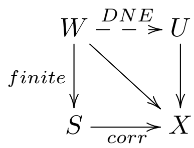
Problem: the Zariski topology does not extend to $SmCorr(k)$. If it were, then for a finite correspondence $c : Y \to X$ and a Zariski cover $U \to X$, then there should exist a Zariski cover $V \to Y$ and a correspondence making a commutative square in $SmCorr(k)$

$$\begin{array}{ccc} V & \longrightarrow & U \\ & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

But this would imply that each time we have a finite correspondence from a local scheme, we have a lifting



and this means that for a finite cover $W \to S$ of a local scheme S as above, we should have a lifting of maps of schemes $W \to U$. But there is no reason why such a lifting should exist.



Definition. The small Nisnevich site on a scheme X is the category of étale morphisms to X and a Nisnevich covering is an étale covering $\{U_i \to X\}$ such that for each point $x \in X$ there exists i and $u \in U_i$ mapping to x such that the induced morphism $k(x) \to k(u)$ is an isomorphism. Alternative, a Nisnevich cover is an étale covering admitting a constructible section.

Definition. Mayer-Vietoris coverings are Nisnevich coverings $\{V \to X, U \to X\}$ such that $V \to X$ is an open immersion and $U \to X$ is an isomorphism outside of V (after taking reduced schemes).

Lemma 13. *The Nisnevich topology is generated by Mayer-Vietoris coverings. The local rings for the Nisnevich topology are the Henselian schemes (for étale we use strictly Henselian).*

Corollary 14. *The Nisnevich topology extends to $SmCorr(k)$.*

$$C_{tr}^{Nis} : DPSh_{tr} \to (I_{tr}^{Nis})^{\perp} \cong D(Sh_{tr}^{Nis})$$
$$F \mapsto (C_{tr}^{Nis})_0(F)(Y) = \prod_{y \in Y} F(Spec\ \mathcal{O}_{Y,y}^h)$$
$$(C_{tr}^{Nis})_1(F) = (C_{tr}^{Nis})_0(Coker(F \to C_{tr}^{Nis})_0(F)))$$

The main theorem is the following.

Theorem 15. *Let k be a perfect field.*

1. $I_{tr}^{\mathbb{A}^1}$ *and* I_{tr}^{Nis} *are compatible (i.e.* $C^{\mathbb{A}^1}C^{Nis}$*).*
2. $I_{tr}^{\mathbb{A}^1,Zar} = I_{tr}^{\mathbb{A}^1,Nis}$ *which implies that* $DM^{eff} = DPSh_{tr}/I^{\mathbb{A}^1,Zar} = DPSh_{tr}/I^{\mathbb{A}^1,Nis} = (I_{tr}^{\mathbb{A}^1})^{\perp} \cap (I_{tr}^{N}is)^{\perp}$.

This theorem is equivalent to the following.

Theorem 16. *Let k be a perfect field and $F \in PSh_{tr}$ an $\mathbb{A}^1$ invariant presheaf. Then*

1. $H^\bullet_{Nis}(X,F) = H^\bullet_{Zar}(X,F)$
2. $H^\bullet_{Nis}(X,F) = H^\bullet_{Nis}(X \times \mathbb{A}^1, F)$.

Rough idea of why (2) implies (2). We want to show that any $\mathbb{A}^1$-local complex of presheaves with transfers which is Zariski local as a mere complex of presheaves is in fact Nisnevich local. It is enough (in fact equivalent) to show that if $F \in DPSh_{tr}$ is $\mathbb{A}^1$ local then $C^{Zar}(F)$ is quasi-isomorphic to $C^{Nis}(F)$. That is, that Zariski hypercohomology is isomorphic to the Nisnevich hypercohomology.

Strategy of the proof. We start with F a presheaf (not assumed to have transfers). We construct $(Cous\ F)^i \in PSh$ and a map $F \to (Cous\ F)^0$. We show that $(Cous\ F)^i$ are Nisnevich sheaves, and that they are flasque.

$$F \to F_{Zar} \to F_{Nis} \to (Cous\ F)^0$$

Suppose now that F has transfers and is $\mathbb{A}^1$-invariant. We define $(Cous\ F)^\bullet$ as a complex. $F_{Zar} \to (Cous\ F)^\bullet$ is a Zariski resolution (hence, $F_{Nis} \to (Cous\ F)^\bullet$ is a Nisnevich resolution).

$$\begin{aligned} H^\bullet_{Nis}(X,F) &= \mathbb{H}^\bullet_{Nis}(X,(Cous\ F)^\bullet) \\ &= H^\bullet((Cous\ F)^\bullet(X)) \\ &= H^\bullet_{Zar}(X,F) \end{aligned}$$

and this proves (b).

2.6 Cousin complex

$(Cous\ F)^i$, $F \in PSh$. $\mathbb{Z}(1)[1] = Cone(\mathbb{Z}_{tr}(\mathbb{G}_m) \to \mathbb{Z}_{tr}(Spec\ k))[-1]$

Definition.

$$F_{-1} = \underline{Hom}_{PSh}(\mathbb{Z}(1)[1], F)$$
$$F_{-1}(X) = Coker(F(X) \to F(X \times \mathbb{G}_m)$$

If $F \in PSh_{tr}$ then F_{-1} is as well. If F is $\mathbb{A}^1$ invariant then F_{-1} is also.

Definition. We define $(Cous\ F)^n(X) = \oplus_{x \in X^{(n)}} F_{-n}(\eta_x)$ where $X^{(n)}$ is the set of points of X of codimension n and $F_{-n} = (F_{1-n})_{-1}$.

Lemma 17. *$(Cous\ F)^i$ is a flasque Nisnevich sheaf*

Cousin differential: Let F be an $\mathbb{A}^1$ invariant presheaf with transfers. We define $d : (Cous\ F)^i \to (Cous\ F)^{i+1}$. The morphism d is an infinite $X^{(n)} \times X^{(n+1)}$-matrix with entries $d_{xy} : F_{-i} \to F_{-i-1}(\eta_y)$. If $y \in \overline{x}$ then $d_{xy} = 0$. If $y \in \overline{x}$ then $d_{xy} = res_y \cdot (F_{-i})(\eta_x) \to (F_{-i-1})(\eta_y)$.

Residue: $y \in \overline{x}$ of codimension one. We define $F(\eta_x) = \mathrm{colim}_{U \subset \overline{x}} F(U) \to F_{-1}(\eta_y) = \mathrm{colim}_{V \subset \overline{y}} F_{-1}(V)$. This means that Z smooth $\subset X$ smooth divisor

$$F(X \backslash Z) \stackrel{rel}{\to} coker(F(Z) \to F(Z \times \mathbb{G}_m))$$

$$coker(F(X) \to F(X \backslash Z)) \stackrel{res}{\to} coker(F(Z \times \mathbb{A}^1) \to F(Z \times \mathbb{G}_m))$$

In classical topology we have tubular neighborhoods $Z \times \mathbb{A}^1 \cong Tub_Z \supset Z \times \mathbb{G}_m \cong Tub_Z \backslash Z$.

$$\begin{CD} Z \times \mathbb{G}_m \cong Tub_Z \backslash Z @>>> X \backslash Z \\ @VVV @VVV \\ Z \times \mathbb{A}^1 \cong Tub_Z @>>> X \end{CD}$$

Definition (Tubular neighborhood/link correspondence in algebraic geometry)**.** Let $Z \subset X$ be a smooth divisor of X smooth. A tubular neighborhood/link correspondence is a commutative diagram of correspondences

$$\begin{CD} \mathbb{Z}_{tr}(Z \times \mathbb{G}_m) @>{\theta}>> \mathbb{Z}_{tr}(X \backslash Z) \\ @VVV @VVV \\ \mathbb{Z}_{tr}(Z \times \mathbb{A}^1) @>{\tilde{\theta}}>> \mathbb{Z}_{tr}(X) \end{CD}$$

such that the specialization of $\tilde{\theta}$ is the identity on Z.

Proposition 18. *Let k be perfect. $X \supset Z$ a smooth pair with X quasi-projective and $P \subset Z$ a set of points.*

1. *There exists $U \subset X$ a Zariski neighborhood of P such that $U \cup Z$ admits a finite correspondence.*

2. *For such a U, θ, θ' be a link correspondence $(U, U \cap Z)$. Then there exists $Z' \subset Z \cap U$ containing P, $\hat{\theta}|_{Z'} = \hat{\theta}'|_{Z'} : \mathbb{Z}_{tr}^{\mathbb{A}^1}(Z')(1)[1] \to \mathbb{Z}_{tr}^{\mathbb{A}^1}(U \backslash Z')$*

We already noticed that $H^\bullet_{Nis}(X, F) \cong H^\bullet_{Zar}(X, F)$ follows from

Theorem 19. 1. *$d^2 = 0$ hence $(Cous\ F)^\bullet$ is a complex-*

2. *$F_{Zar} \to (Cous\ F)^\bullet$ is a resolution of F_{Zar}.*

We need to show that $p_X^* : (Cous\ F)^\bullet(X) \cong (Cous\ F^\bullet)(X \times \mathbb{A}^1)$ is an isomorphism. We show that it is a filtered quasi-isomorphism.

$G \subset (Cous\ F)(X) \to (G')^i \subset Cous\ F(X \times \mathbb{A}^1)$ Stupid filtration.

G^i/G^{i+1} in degree i $\oplus_{x \in X, \mathrm{codim}\ x = i} F_{-i}(\eta_x)$

$(G')^i =$ something illegible.

To finish the proof we need to prove the following lemma.

Lemma 20. *$F \in PSh_{tr}$ $\mathbb{A}^1$-invariant. η a generic point of a smooth variety. Then $F(\mathbb{A}^1_\eta) \cong H^0_{Zar}(\mathbb{A}^1_\eta, F)$ and $H^1_{Zar}(\mathbb{A}^1_\eta, F) = 0$.*

3 Homotopy theory of schemes I

We begin with some analogies between classical topology and motivic topology.

TOPOLOGY	MOTIVIC TOPOLOGY
Spaces	Schemes
$[0,1]$	$\mathbb{A}^1$
Topological K-theory	K-theory
$D(Ab)$	Motives

Fix a base scheme S, that should be separated, regular, noetherian, and finite dimensional. We will be interested in the category of smooth schemes Sm/S over S. For $X \in Sm/S$, are there any interesting homotopy types associated with X? Rather than going into details of a construction we discuss some vague principles. For example, Mayer-Vietoris. If we apply K-theory to a Zariski cover $\{U \to X, V \to X\}$, we get a homotopy push-out diagram.

$$\begin{array}{ccc} K(X) & \longrightarrow & K(V) \\ \downarrow & & \downarrow \\ K(U) & \longrightarrow & K(U \cap V) \end{array}$$

This was shown by Brown and Gersten in 73. The second thing we would like to incorporate into our homotopy theory of schemes is homotopy. If we have two maps $f, g : X \to Y$ we would like to define them to be homotopic if there exists a map $\mathbb{A}^1 \times X \to Y$ with the appropriate property. This leads to the notion of an elementary $\mathbb{A}^1$-homotopy equivalence. Another problem is the non-existence of colimits in Sm/S. We want to link the two categories

1. Sm/S which contains for example $\mathbb{A}^n, \mathbb{G}_m, \mathbb{P}^n$, and

2. $sSet$ simplicial sets which contains the spheres S^n and simplexes $\Delta[n]$.

There are two obvious functors

$$Sm/S \to sPre(Sm/S) \leftarrow sSet$$

towards the category of simplicial presheaves on Sm/S (the Yoneda embedding and the constant presheaf functor). We then define the a model structure on $sPre(Sm/S)$.

Definition. A map $X \to Y$ of simplicial presheaves is called a weak equivalence / weak fibration if $X(Z) \to X(Z)$ is a weak equivalence / weak fibration for all $Z \in Sm/S$.

These definitions lead to a model structure. We think of this as a "free" model structure on Sm_S. The next step is to impose relations. There are two sources for relations.

1. Mayer-Vietoris, and
2. $\mathbb{A}^1$-invariance.

We choose to use the Nisnevich topology for various reasons:

1. $H^p_{Nis}(X, M) = 0$ for $p > \dim(X)$.
2. K-theory satisfies Nisnevich descent.
3. A regular closed embedding $Z \subset X$, is isomorphic (Nisnevich locally) to $\mathbb{A}^e \subset \mathbb{A}^e \times \mathbb{A}^c$ (where $e = dim\ Z$ and $e + c = dim\ X$.

Definition. A Nisnevich distinguished square (NDS) is a cartesian square of the form

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

where $U \subset X$ is an open immersion and $V \to X$ is an étale map which is an isomorphism outside of U.

Another reason to use the Nisnevich topology is that

4. Every NDS is a pushout in $Shv(Sm/S)_{Nis}$.

If we have a Zariski covering $\{U \to X, V \to X\}$ the corresponding square is not a push-out square in the category of presheaves.

To impose relations on the "free" model category structure we will use Bousfield localizations.

Recall: Let C be a set of maps in a (nice) model category $\mathcal{M}$. A Bousfield localization is a map $\mathcal{M} \to L_C\mathcal{M}$ of model categories that satisfies the following universal property. For a morphism of model categories $\mathcal{M} \to \mathcal{N}$ such that every morphism in C is sent to a weak equivalence, there exists a unique extension.

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & L_C\mathcal{M} \\ & \searrow & \swarrow \\ & \mathcal{N} & \end{array}$$

$L_C\mathcal{M}$ is called the Bousfield localization of $\mathcal{M}$ with respect to C.

Definition. The motivic model structure on $sPre(Sm/S)_{Nis}$ is the Bousfield localization of the free model category with respect to

1. $hocolim(U \leftarrow W \to V) \to X$ for each NDS,
2. $Z \times \mathbb{A}^1 \to Z$ for every $Z \in Sm/S$.

Recall that homotopy colimits and colimits are not the same. $colim(* \leftarrow S^n \rightarrow *)$ is contractible but $hocolim(* \leftarrow S^n \rightarrow *)$ is S^{n+1}.

Comments.

1. The motivic cofibrations are the same as the free cofibrations.
2. X is free motivic fibrant if and only if it is free fibrant and satisfies Nisnevich descent, and is $\mathbb{A}^1$-invariant.
3. There is a Quillen equivalence $sPre(Sm/S)_{Nis} \leftrightarrows Shv(Sm/S)_{Nis}$.

Some examples of motivic weak equivalences are the following.

1. $\mathbb{A}^1 \rightarrow S$.
2. Elementary $\mathbb{A}^1$-homotopy equivalences.
3. Vector bundles $E \rightarrow X$ (Hint: take an affine cover of X over which the bundle trivializes).

Definition. $MS_\bullet$ is the category of pointed motivic spaces. That is, the category of simplicial presheaves of pointed simplicial sets.

In $MS_\bullet$ there are weak equivalences $T = \mathbb{A}^1/\mathbb{G}_m \sim S^1 \wedge \mathbb{G}_m \sim (\mathbb{P}^1, \infty)$. To see this, we consider

$$\begin{array}{ccccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 & \longrightarrow & \\ \downarrow & & & & \\ \Delta[1] \wedge (\mathbb{A}^1, 0) & \longrightarrow & X & & \end{array}$$

We note that $\Delta[1] \wedge (\mathbb{A}^1, 0)$ is contractible so $X \sim \mathbb{A}^1/\mathbb{G}_m$. On the other hand, $\mathbb{A}^1$ is contractible so $X \sim S^1 \wedge \mathbb{G}_m$. Lastly, we have a NDS $\{\mathbb{A}^1 \rightarrow \mathbb{P}^1, \mathbb{A}^1 \rightarrow \mathbb{P}^1\}$ which shows that $X \sim (\mathbb{P}^1, 0)$.

Thom spaces: Let $E \rightarrow X$ be a vector bundle. We define the Thom space as $E/(E \backslash X)$. A map of vector bundles gives rise to a morphism if Thom spaces. And a morphism of schemes $Y \rightarrow X$ gives a morphism $Th(f^*E) \rightarrow Th(E)$.

In $Ho(MS_\bullet)$ there are isomorphisms $Th(\mathcal{O}_X^n) \cong T^{\wedge n} \wedge X_+$ and $Th(E) \cong \mathbb{P}(E \oplus \mathcal{O})/\mathbb{P}(E)$.

$$\begin{array}{ccc} E \backslash X & \longrightarrow & E \\ \downarrow & & \downarrow \\ \mathbb{P}(E \oplus \mathcal{O}) \backslash X & \longrightarrow & \mathbb{P}(E \oplus \mathcal{O}) \end{array}$$

is a NDS so we have $Th(E) \cong \mathbb{P}(E \oplus \mathcal{O})/(\mathbb{P}(E \oplus \mathcal{O}) \backslash X)$.

References

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