Motives and Milnor conjecture

notes by S. Kelly:

Day II

Contents

1 Motivic complexes and motivic cohomology III

Recall: we have defined motivic cohomology by analogy with classical algebraic topology

$$H^n_M(X,\mathbb{Z}(q)) \cong \hom_{D^-PSh_{tr}}(\mathbb{Z}_{tr}(X), C^{\mathbb{A}^1}(\mathbb{Z}_{tr}(\mathbb{P}^{d-1})/\mathbb{Z}_{tr}(\mathbb{P}^d)[-2d])[n])$$

and asked if this satisfies a Mayer-Vietoris long exact sequence. We also defined

$$DM^{eff} = D^-PSh_{tr}/\left\langle \begin{array}{c} \mathbb{Z}_{tr}(U\cap V)\to\mathbb{Z}_{tr}(U)\oplus\mathbb{Z}_{tr}(V)\to\mathbb{Z}_{tr}(U\cup V) \\ Cone(\mathbb{Z}_{tr}(\mathbb{A}^1\times X)\to\mathbb{Z}_{tr}(X)) \end{array}\right\rangle$$

Theorem 1 (Main theorem of the lecture (due to Voevodsky))**.** *If k is perfect then $DM^{eff} = \mathbb{A}^1$ local objects in DSh^{Nis}_{tr}.*

We also have

Theorem 2. *Let k be a perfect field and $F \in PSh_{tr}$ an $\mathbb{A}^1$ invariant presheaf. Then*

1. $H^\bullet_{Nis}(X,F) = H^\bullet_{Nis}(X\times\mathbb{A}^1,F)$.
2. $H^\bullet_{Nis}(X,F) = H^\bullet_{Zar}(X,F)$

The strategy of the proof is

$$F \to F_{Zar} \to F_{Nis} \to (Cous\ F)^{\bullet}$$

where $(Cous\ F)^i$ is a flasque Nisnevich sheaf, $(Cous\ F)^i = \oplus_{x \in X^{(i)}} F_{-i}(\eta_x)$ and the differential d comes from

$$res : F(\eta) \to F_{-1}(s)$$

for T a semilocal scheme, η is the union of the generic points and s the disjoint union of the closed points. The existance of residues comes from the existance of *link correspondances*

Definition (Link correspondance)**.** Let $Z \subset X$ be a smooth divisor of X smooth. A link correspondence is a commutative diagram of correspondences

$$\begin{array}{ccc} \mathbb{Z}_{tr}(Z \times \mathbb{G}_m) & \xrightarrow{\theta} & \mathbb{Z}_{tr}(X \backslash Z) \\ \downarrow & & \downarrow \\ \mathbb{Z}_{tr}(Z \times \mathbb{A}^1) & \xrightarrow{\tilde{\theta}} & \mathbb{Z}_{tr}(X) \end{array}$$

such that the specialization of $\tilde{\theta}$ is the identity on Z.

Proposition 3. *Let k be perfect. $X \supset Z$ a smooth pair with X quasi-projective and $P \subset Z$ a set of points.*

1. *There exists $U \subset X$ a Zariski neighborhood of P such that $(U, U \cap Z)$ admits a link correspondence.*
2. *If θ, θ' are two such link correspondences for $(U, U \cap Z)$, then there exists $Z' \subset Z \cap U$ containing P such that $\hat{\theta}|_{Z'} = \hat{\theta}|_Z : \mathbb{Z}_{tr}^{\mathbb{A}^1}(Z')(1)[1] \to \mathbb{Z}_{tr}^{\mathbb{A}^1}(U \backslash Z')$*

Proposition 4.

1. *$d^2 = 0$ hence $(Cous^{\bullet})F$ is a complex,*
2. *$F_{Zar} \to (Cous\ F)^{\bullet}$ is a resolution.*

We reduced to the following lemma.

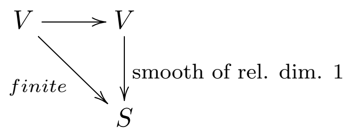
Lemma 5. *Let $F \in PSh_{tr}$ be an $\mathbb{A}^1$ invariant presheaf and η a generic point of a smooth variety. Then*

1. *$F(\mathbb{A}^1) \cong H^0_{Zar}(\mathbb{A}^1_{\eta}, F)$*
2. *$H^1_{Zar}(\mathbb{A}^1_{\eta}, F) = 0$.*

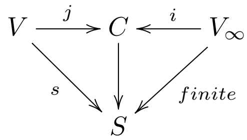
1.1 Tubular neighbourhood proof of proposition 3 and lemma 5

1. In proposition 3 we can assume k infinite.
2. k infinite implies a nice geometric reduction of proposition 3.

Definition (Standard family of curves)**.** Let V, S be smooth affine schemes and suppose we have the following diagram.



1. There exists a diagram

   with j an open embedding, i its closed complement, C proper over S, and $T = V_\infty \cup Z$ has an affine neighbourhood in C.

2. $\Lambda_Z : Z \to Z_V$ is a principle divisor

For example, $U \to V \leftarrow \{z\}$ where V is a smooth affine curve, z a k-point such that $z = div\ \phi$ for $\phi \in \theta(V)$.

Proposition 6. *Let k be an infinite field, X quasi-projective and smooth, $Z \subset X$ a closed embedding, $P \subset X$ a finite set of points. Then P admits a nieghbourhood $V \subset X$ such that there exists S with $(V/Z, Z/S)$ a nice family, and if Z is a divisor which is smooth at $Z \cap P$ then we can choose $Z \to S$ étale.*

Proof. Bertini type. □

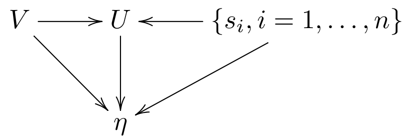
Proposition 7. *Let $(V/S, Z)$ be a standard family of curves.*

1. $\mathbb{Z}_{tr}^{\mathbb{A}^1}(V\backslash Z) \underset{\phi}{\longmapsto} \mathbb{Z}_{tr}^{\mathbb{A}^1}(V)$ *with* $\phi \in Corr(V, V\backslash Z)$ *a right inverse to* j $\phi \in \Gamma(Z_V, \mathcal{O}_{Z_V})$, $1_Z(Z) = div\ \phi$.
2. *if every component of Z surjects to a component of S then $\mathbb{Z}_{tr}^{\mathbb{A}^1}(V\backslash Z/V)$ is independant of V (for Z/S fixed).*
3. *if moreover $Z \to S$ is étale $\mathbb{Z}_{tr}^{\mathbb{A}^1}(V\backslash Z/V)\ left\tilde{a}rrow\ \mathbb{Z}_{tr}^{\mathbb{A}^1}(Z)(1)[1]$. canonical given by link correspondance.*

Lemma 8. *Let $F \in PSh_{tr}$ be an $\mathbb{A}^1$ invariant presheaf and η a generic point of a smooth variety, $U \subset \mathbb{A}^1_\eta$ an open. Then*

1. $F(U) \cong F_{Zar}(U) \cong F_{Nis}(U)$
2. $H^{>0}_{Zar}(U, F) = H^{>0}_{Nis}(U, F) = 0.$

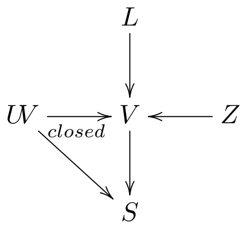
Proof.



a standard family form proposition 1', $\mathbb{Z}^{\mathbb{A}^1}_{tr}(U) = \mathbb{Z}^{\mathbb{A}^1}_{tr}(V) \oplus \mathbb{Z}^{\mathbb{A}^1}_{tr}(Z)(1)[1]$

$$0 \to F(V) \to F(U) \to \oplus F_{-1}(s_1) \to 0$$

Then F is a Zariski sheaf and does not have any sections supported on closed points which implies that $F(U) = R\Gamma(U_{Zar}, F)$.



Nisnevich covering $L|_Z = Z$ and so $\mathbb{Z}^{\mathbb{A}^1}_{tr}(L\backslash Z/Z) \cong \mathbb{Z}^{\mathbb{A}^1}_{tr}(V\backslash Z/Z)$. □

1.2 Construction of correspondances for such families

$$c : Pic(C, T) \to Cor^{\mathbb{A}^1}(S, V\backslash Z)$$

where $Pic(C, T)$ is the group of isomorphism classes of pairs $(\mathcal{L}, \lambda)$ where $\mathcal{L}$ is a line bundle on C and λ is a trivialisation on T and $Cor^{\mathbb{A}^1}$ are finite correspondances up to homotopy.

$(\mathcal{L}, \lambda) \to$ affine space of regular sections $\tilde{\lambda}$ of $\mathcal{L}$ in a neighbourhood of T (depending on λ) with $\tilde{\lambda}|_T = \lambda^{-1} : \mathcal{O}_T \cong L_T$.

$\tilde{\lambda}$ extends to a meromorphic section of $\mathcal{L}$ on C.

$div\ \tilde{\lambda} \in Cor(S, C\backslash T = V\backslash Z)$ because $div\ \tilde{\lambda}$ is quasi affine proper over S so $div\ \tilde{\lambda} \to S$ is finite $\leftarrow c(\mathcal{L}, \lambda)$ class of $\tilde{\lambda}$.

If $(C'/S, T')$ is another family

$$c : Pic(C_{C'\backslash T'}, T_{C'\backslash T'}) \to Cor^{\mathbb{A}^1}(C'-T', (C-T)_{C'-T'}) \to Cor^{\mathbb{A}^1}(C'-T', (C-T))$$

Example 9. We take $C' = C$ and $T' = T$ and take the identity correspondance $C - T \to C - T$. This is just $c(\mathcal{O}_{C_{C-T}}(\Delta_{C\backslash T}), 1_{T_{C\backslash T}}$ where $C\backslash T \to C_{C-T}$.

Proof of proposition 3.

1. $\mathbb{Z}_{tr}^{\mathbb{A}^1}(V\backslash Z) \xrightarrow[s_\phi]{} \mathbb{Z}_{tr}^{\mathbb{A}^1}(V)$ where $\Delta_Z(Z) = div\ \phi$, $\phi \in \mathcal{O}(Z_V)$.

$$c : Pic(C_V, T_V) \to Cor^{\mathbb{A}^1}(V, C_V\backslash T_V) \to Cor^{\mathbb{A}^1}(V, V\backslash Z)$$

$$\theta_{C_V}(\Delta_V(V)), \hat{\phi}^{-1} \mapsto \Delta_p$$

where $\Delta_V : V \to C_V$ is the diagonal, $\hat{\phi}^{-1}$ trivialisation of $\mathcal{O}_{C_V}(\Delta_V(U))$ on $T_V = (V^\infty)_V \amalg Z_V$, $\hat{\phi}^{-1} = 1 + \phi^{-1}$. This is a splitting.

$$(V - Z) \to V \to V - Z = (c(\mathcal{O}_{C_{V-Z}}(\Delta_{V\backslash Z}(V\backslash Z))), 1) = id_{V\backslash Z}$$

2. $\mathbb{Z}_{tr}^{\mathbb{A}^1}(V\backslash Z/V)$ is independant of V (for Z/S fixed). Let (V'/S) be another standard family. Choose $\phi' \in \mathcal{O}(Z_{V'})$ trivialising $Z \to Z_V$

$$\pi_{VV'}^{\phi'} = c(\mathcal{O}_{C_{V'\backslash Z}}, \hat{\phi}' = 1 + phi')$$

$$\begin{array}{ccc} \mathbb{Z}_{tr}^{\mathbb{A}^1}(V'\backslash Z) & \longrightarrow & \mathbb{Z}_{tr}^{\mathbb{A}^1}(V\backslash Z) \\ & \searrow^{0=c(\mathcal{O}_{C_{V'\backslash Z}},1)} & \downarrow \\ & & \mathbb{Z}_{tr}^{\mathbb{A}^1}(V) \end{array}$$

$\mathbb{Z}_{tr}^{\mathbb{A}^1}(V' - Z/V') \cong \mathbb{Z}_{tr}^{\mathbb{A}^1}(V - Z/V)$. Claim is that the isomorphism is independant of ϕ'. It is enough to show that $\pi_{VV''}^{\phi''} = \pi_{VV'}^{\phi'}\pi_{V'V''}^{\phi''}$ which is an exercise on divisors.

3. $Z \to S$ étale. $\mathbb{Z}_{tr}^{\mathbb{A}^1}(Z)(1)[1] \cong \mathbb{Z}_{tr}^{\mathbb{A}^1}(V\backslash Z/V)$ $V' = Z \times_S \mathbb{A}^1_S$ Claim: $Z \to Z_{V'}$ is principle.

□

Proof. We want to show that $d^2 = 0$ and that $F_{Zar} \to (Cous\ F)^\bullet$ is a resolution. We cna once more assume that k is infinite by a Gaois argument and then use some devissage.

Proof that $d^2 = 0$. This is a local statement and so we can assume that X is affine. We pick $\phi \in (cous^n\ F)(X)$ and what to show that $d^2\phi = 0$. We choose $Y \subset X$ the support of ϕ and let ξ be the union of the generic points of the support of $d^2\phi$. Since X is affine, we can find an embedding in some affine space $X \to T$ and a projection $T \to L$ such that $Y \to L$ is finite, and that $\xi \to L$ is an embedding (this is a Bertini type argument).

We claim that $d^2\phi = d^2(tr_{Y/L}\phi)$. This comes from the fact that d commutes with g_* for g proper. So we have reduced to the case of a vector space and can assume that $X = L$ and that $\phi \in (Cous\ F)^0(L) = F(\eta_L)$. We take

$$Y_Z \subset Y_1 \subset L$$

each inclusion of codimension 1. $\tilde{\phi} \in F(L \backslash Y_1) \to F(\eta) \ni \phi$.

$$\tilde{d\phi} \in F_{-1}(Y_1 \backslash Y_2) \to \tilde{d\phi} \in F_{-1}(\eta_{Y_1})$$

$L = \mathbb{A}^1 \times D$ with $Y_1 \to D$ finite, and $\eta_{Y_2} \to D$ an embedding.

$$d^2\phi = d(tr_{Y_1/D}(d(\phi)) \in F_{-2}(\eta)$$

$L = \mathbb{A}^1 \times D \subset \overline{L} = \mathbb{P}^1 \times D$ $d(\phi = d(\phi + Res_\infty \phi$. The fact that d commutes with traces gives

$$tr_{Y_1 \backslash D} d\phi + res_\infty \phi = 0 \in F_{-1} D$$

but then $d^2\phi = dRes_\infty \phi$ $Res_\infty(\phi$ restructs to the generic point of D of $Res(\tilde{\phi}) \in F_{-1,Zar} D$ Hence $Res_\infty \phi = 0$. □

2 Homotopy theory of schemes II

Recall that we stated homotopy purity yesterday.

Theorem 10 (Homotopy purity)**.** *Let $Z \to X$ be a closed embedding of smooth schemes. Then in the homotopy category there is an isomorphism*

$$Th(N_{X,Z}) \cong X/(X - Z)$$

where $N_{X,Z}$ is the normal bundle to Z in X.

Outline of the proof. Step 1: We have a diagram

$$\begin{array}{ccccc} Bl_{Z\times\{0\}}(X \times \mathbb{A}^1) & \xrightarrow{p} & X \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^1 \\ & \nwarrow^{f} & \uparrow & & \\ & & Z \times \mathbb{A}^1 & & \end{array}$$

and $\pi^{-1}(\mathbb{G}_m) = X \times \mathbb{G}_m$ and $\pi^{-1}(0) = \mathbb{P}(N_{X,Z} \oplus \mathcal{O}_Z) \amalg Bl_Z(X)$. The fiber of the pair $(Bl_{Z\times\{0\}}(X \times \mathbb{A}^1), f(Z \times \mathbb{A}^1))$ over 1 is (X, Z). This gives us a map

$$Th_1 : X/(X - Z) \to Bl_{Z\times\{0\}}(X \times \mathbb{A}^1)/(Bl_{Z\times\{0\}}(X \times \mathbb{A}^1) - f(Z \times \mathbb{A}^1))$$

Over 0 we have

$$\begin{array}{ccc} Bl_{Z\times\{0\}}(X \times \mathbb{A}^1) & \longleftarrow & \mathbb{P}(N_{X,Z} \oplus \mathcal{O}_Z) \\ {\scriptstyle f_0}\uparrow & & \uparrow \\ Z & \xrightarrow[\text{0 section}]{} & N_{X,Z} \end{array}$$

$$p^{-1}(Z \times \{0\}) = \mathbb{P}(N_{X,Z} \oplus \mathcal{O}_Z)$$
$$p^{-1}(Z \times \{0\}) \cap f(Z \times \mathbb{A}^1) = f(Z \times 0)$$

$$Th_2 : Th(N_{X,Z}) \xrightarrow{\sim} p^{-1}(Z \times 0)/(p^{-1}(Z \times 0) - f(Z \times 0))$$
$$\to Bl_{Z\times\{0\}}(X \times \mathbb{A}^1)/(Bl_{Z\times\{0\}}(X \times \mathbb{A}^1) - f(Z \times \mathbb{A}^1))$$

Step 2: [Gro67, 17.12.2d] There exists an open affine covering $\{U_i\}$ of X tocether with étale maps $\alpha_i : U_i \to \mathbb{A}^n$ such that $\alpha_i^{-1}(\mathbb{A}^{n-c} \times 0) = Z \cap U_i$ for some c.

We can make a simplicial presheaf $\mathcal{X}_\bullet$ by $\mathcal{X}_n = (\amalg U_i)_X^{n+1}$ and there exists a motivic weak equivalence $\mathcal{X} \xrightarrow{\sim} X$ [MV99, Lemma 2.11]. Define $\mathcal{Z}_\bullet$ by $\mathcal{Z}_n = (\amalg U_i \cap Z)_X^{n+1}$. Apply Thom spaces and blow-ups degreewise and we obtain a commutative diagram

$$\begin{array}{ccccc}
\mathcal{X}_\bullet/(\mathcal{X}_\bullet - \mathcal{Z}_\bullet) & \xrightarrow{Th_1} & Bl_{\mathcal{Z}_\bullet\times\{0\}}(\mathcal{X}_\bullet \times \mathbb{A}^1)/(etc) & \longleftarrow & Th(N_{\mathcal{X}_\bullet,\mathcal{Z}_\bullet}) \\
\downarrow & & \downarrow & & \downarrow \\
X/(X - Z) & \longrightarrow & Bl_{Z\times\{0\}}(X \times \mathbb{A}^1)/(etc) & \longleftarrow & Th(N_{X,Z})
\end{array}$$

Exercise 11. Vertical maps are motivic weak equivalences.

$$\begin{array}{ccccc}
\mathcal{X}_\bullet & \longleftarrow & \mathcal{X}_\bullet - \mathcal{Z}_\bullet & \longrightarrow & * \\
\downarrow{\scriptstyle \text{motivic w.e.}} & & \downarrow{\scriptstyle \text{motivic w.e.}} & & \| \\
X & \longleftarrow & X/Z & \longrightarrow & *
\end{array}$$

Step 3: By step 2 we may assume that there exist étale maps $\alpha : X \to \mathbb{A}^n$ such that $\alpha_i^{-1}(\mathbb{A}^{n-c} \times 0) = Z$ for some c. So we have a new form $X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^c)$ and morphisms $\alpha \times id_{\mathbb{A}^c} : Z \times \mathbb{A}^c \to \mathbb{A}^n$. Its fiber over $\mathbb{A}^{n-c} \times 0$ is the closed subscheme $Z \times_{\mathbb{A}^{n-c}} Z$. The fact that $Z \to \mathbb{A}^{n-c}$ is étale implies that $\Delta : Z \to Z \times_{\mathbb{A}^{n-c}} Z$ is open. It is also closed. Hence $Z \times_{\mathbb{A}^{n-c}} Z = im(\Delta_Z) \amalg Y$ where Y is a closed subscheme.

So far we have constructed

$$\begin{array}{ccc}
X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^c) & \longrightarrow & Z \times \mathbb{A}^c \\
\downarrow & & \downarrow \\
X & \longrightarrow & \mathbb{A}^n
\end{array}$$

where all maps are étale.

We define $U = X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^c) - Y$. From the diagram we have étale projection maps

$$pr_1 : U \to X$$
$$pr_2 : U \to \mathbb{A}^c$$

By construction $pr_1^{-1}(Z) = Z$ while $pr_2^{-1}(Z \times 0) = Z \times 0$. So there exists a commutative diagram

$$\begin{array}{ccccc}
X/(X-Z) & \xrightarrow{Th_1} & Bl_{Z\times\{0\}}(X\times\mathbb{A}^1)/(etc) & \xleftarrow{Th_2} & Th(N_{X,Z}) \\
\uparrow & & \uparrow & & \uparrow \\
U/(U-pr_1^{-1}Z) & \xrightarrow{Th_1} & etc & \longleftarrow & Th(N_{U,pr_1^{-1}(Z)}) \\
\downarrow & & \downarrow & & \downarrow \\
Z\times\mathbb{A}^c/(Z\times\mathbb{A}^c - Z\times 0) & \xrightarrow{Th_1} & etc & \longleftarrow & Th(N_{Z\times\mathbb{A}^1, Z\times 0})
\end{array}$$

Notice that this does not work Zariski locally. All the vertical maps are motivic weak equivalences.

Step 4: By step 3 we may assume that $Z \to X$ is nothing but the zero section of a trivial vector bundle. Hint: $Bl_{Z\times\{0\}}(Z \times \mathbb{A}^c)$ is the total space of the canonical line bundle over $\mathbb{P}^c_Z$, hence a motivic weak equivalence. □

2.1 How to "compute" maps in the homotopy category $H(MS_\bullet)$

We start with a basic fact. If $\mathcal{M}$ is a model category and we consider $H(\mathcal{M}) = \mathcal{M}[w.e.^{-1}]$ its homotopy category, then $\hom_{H(\mathcal{M})}(A, B) = \hom_{\mathcal{M}}(A, B)/$ homotopy relation, provided A is cofibrant and B fibrant.

In $MS_\bullet$ the homotopy relation is the usual one, where we can use either $\mathbb{A}^1$ or the simplicial unit interval Δ^1.

Example 12. Recall that for every smooth scheme X the object X_+ is cofibrant and also $L \wedge X_+$ is cofibrant for any pointed simplicial set L. We then define the K-theory simplicial presheaf

$$\mathcal{K} : Sm_S^{op} \to sSet_\bullet$$

This associates to X the simplicial set $\Omega(NQVect(X))^f$ where $Vect(-)$ is the category of vector bundles over X, Q is Quillens Q-construction, N is the nerve, $(-)^f$ is fibrant replacement in the category of simplicial sets and Ω is the loop space. We recall that the nerve of a category has as n-simplicies the set of n-tuples of composable morphisms $\to\to\cdots\to$. The loop space Ω of a simplicial set is the internal hom $\underline{\hom}_{sSet_\bullet}(S^1, L)$.

The object $\mathcal{K}$

1. is "free" fibrant,

2. sends NDSs to homotopy pull-back squares (Thomas),

3. is homotopy invariant $\mathcal{K}(X \times \mathbb{A}^1) \cong K(X \times \mathbb{A}^1)$ (Quillen).

$$\begin{aligned}\hom_{H(MS_\bullet)}(S^n \wedge X, \mathcal{K}) &= \hom_{MS_\bullet}(S^n \wedge X_+, \mathcal{K})/\sim \\ &= \hom_{sSet_\bullet}(S^n, \mathcal{K}(X)) \\ &= \pi_n(\mathcal{K}(X)) \\ &= K_n(X)\end{aligned}$$

Recall in the talk BI we had the definition of motivic cohomology as

$$H^n_M(X, \mathbb{Z}(d)) \cong \hom_{D-PSh_{tr}}(\mathbb{Z}_{tr}(X)[2d], C^{\mathbb{A}^1}(\frac{\mathbb{Z}_{tr}(\mathbb{P}^d)}{\mathbb{Z}_{tr}(\mathbb{P}^{d-1})})[n])$$

where $S = Spec\ k$ and k is perfect. Using the fact that the target is $\mathbb{A}^1$ local we can compute this is

$$\hom_{DM^{eff}}(\mathbb{Z}_{tr}(X)[2d], C^{\mathbb{A}^1}(\frac{\mathbb{Z}_{tr}(\mathbb{P}^d)}{\mathbb{Z}_{tr}(\mathbb{P}^{d-1})})[n])$$

We can shift this to the homotopy category using some adjunctions. There is a functor

$$MS_\bullet = Fun(Sm/k^{op}, sSet_\bullet) \stackrel{\mathbb{Z}_{tr}}{\to} AddFun(SmCorr(k)^{op}, sAb)$$

and we have $\mathbb{Z}_{tr}(X_+) = \mathbb{Z}_{tr}(X)$ and $\mathbb{Z}_{tr}(L) = \mathbb{Z}(L)/\mathbb{Z}*$ for $L \in sSet_\bullet$. We have the Dold-Kan correspondance

$$AddFun(SmCorr(k)^{op}, sAb) \cong AddFun(SmCorr(k)^{op}, Ch_{\geq 0})$$

and a further obvious functor

$$AddFun(SmCorr(k)^{op}, Ch_{\geq 0}) \to DPSh_{tr}/I^{\mathbb{A}^1, Nis}.$$

Putting these together we make a commutative diagram

$$\begin{array}{ccc} MS_\bullet & \longrightarrow & AddFun(SmCorr(k)^{op}, sAb) \\ \Big\downarrow & & \Big\downarrow \text{Dold-Kan} \\ & & AddFun(SmCorr(k)^{op}, Ch_{\geq 0}) \\ & & \Big\downarrow \\ H(MS_\bullet) & \longrightarrow & DPSh_{tr}/I^{\mathbb{A}^1, Nis} \end{array}$$

2.2 Motivic stable homotopy theory

$$\pi_3(S^2) \longrightarrow \pi_4(S^3) \stackrel{\sim}{\longrightarrow} \pi_5(S^4) \stackrel{\sim}{\longrightarrow} \ldots$$

Hopf map. This is a first example of the Freudenthal suspension theorem, which says that homotopy groups become stable as we suspend. To capture stable phenomena we replace spaces by spectra and in the new homotopy category - the stable homotopy category - the suspension becomes an invertible operation.

Definition. A motivic T-spectrum (where T is an object of $MS_\bullet$) is a sequence of objects $\{E_n \in MS_\bullet, n = 0, 1, \dots\}$ together with maps $\sigma : T \wedge E_n \to E_{n+1}$ called assembly morphisms. A morphism of spectra is a sequence of maps $E_n \to F_n$ such that the squares commute.

$$\begin{array}{ccc} T \wedge E_n & \longrightarrow & E_{n+1} \\ \downarrow & & \downarrow \\ T \wedge F_n & \longrightarrow & F_{n+1} \end{array}$$

The category of motivic T-spectra is denoted $Spt_T(MS_\bullet)$.

Example 13. Let $\mathcal{X}$ be an object of $MS_\bullet$. We define a spectrum $\Sigma_T^\infty \mathcal{X}$ that has $(\Sigma_T^\infty \mathcal{X})_n = T^{\wedge n} \wedge \mathcal{X}$ with the obvious identities $T \wedge T^{\wedge n} \wedge \mathcal{X} \to T^{\wedge n+1} \wedge \mathcal{X}$ as assembly morphisms. The sphere spectrum, denoted $\mathbb{1}$ is $\mathbb{1} = \Sigma_T^\infty S_+$ where S is the base scheme.

We define many model structures on the category of motivic spectra. To begin with, the level-wise structure. The level-wise weak-equivalences and level-wise fibrations are the morphisms $f_\bullet : E_\bullet \to F_\bullet$ such that each f_n is a weak-equivalence or fibration in $MS_\bullet$. There always exists a fibrant replacement $E \to E^{fib}$ for any motivic T-spectra. We define

$$QE_n^{fib} = colim(E_n^{fib} \to \Omega_T E_{n+1}^{fib} \to \Omega_T^2 E_{n+2}^{fib} \to \dots)$$

where Ω_T is the inverse to $T \wedge -$ on the homotopy category of spectra. We have morphisms $E \to E^{fib} \to QE^{fib}$.

For the stable model structure on the category of spectra, a morphism is a stable weak equivalence if $QE^{fib}(X) \to QF^{fib}(X)$ is a weak equivalence for each $X \in Sm/S$. The cofibrations are the morphisms $f_\bullet : E_\bullet \to F_\bullet$ such that the morphisms $E_{n+1} \amalg_{T \wedge E_n} T \wedge F_n \to F_{n+1}$ are cofibrations.

A spectra E is stably fibrant if and only if it is level-wise fibrant and the adjoints of the assembly morphims $E_n \to \Omega_T E_{n+1}$ are weak equivalences in $MS_\bullet$ (in the litterature these are often called Ω-spectra).

Theorem 14 (Jardine)**.** *The above defines a model structure.*

2.3 Cogent properties of $SH(S)$ (the motivic stable homotopy category).

1. The category $SH(S)$ is triangulated.

   (a) The distinguished triangles are

   $$E \xrightarrow{f} F \to Cone(F) \to E[1]$$

   where for a morphism $f : E \to F$ we define $Cone(f) = colim(F \xleftarrow{f} E \to E \wedge \Delta[1])$.

(b) The functor $T\wedge -$ is a self-equivalence of $SH(S)$ (where $T = S^1 \wedge \mathbb{G}_m$).

(c) To define the addition of two morphisms $\alpha, \beta : E \to F$ we use

$$E \stackrel{\Delta}{\to} E \times E \stackrel{\cong}{\leftarrow} E \vee E \stackrel{\alpha \vee \beta}{\to} F$$

2. We define the mixed motivic spheres as $S^{p,q} = S^{p-q} \wedge \mathbb{G}_m^q$ and for a motivic spectrum E we define homology and cohomology with coefficients in E as respectively

$$E_{p,q}(X) = \hom_{SH}(S^{p,q}, E \wedge X_+)$$
$$E^{p,q}(X) = \hom_{SH}(X_+, E \wedge S^{p,q})$$

3. K-theory and motivic cohomology are represented in this way.

4. SH is symmetric monoïdal and the unit for the tensor product is the sphere spectrum $\mathbb{1}$.

5. There exist other categories with closed symmetric monoïdal model structures with homotopy categories equivalent to SH. For example motivic symmetric spectra (roughly spectra equipped with actions by symmetric groups) and motivic functors.

2.4 Algebraic cobordism

For $n \geq d \geq 0$ we have a total space $\gamma_{d,n} : T(d,n) \to Gr(d,n)$ over the Grassmanian. These can be thought of as schemes or functors. As functors on affine schemes we have

$$Gr(d,n)(R) = \left\{ M \subset R^n \mid M \begin{array}{c} \text{is a finitely generated projective} \\ \text{submodule of rank } d \text{ locally split} \end{array} \right\}$$

$$T(d,n)(R) = \{(M,V) \subset Gr(d,n)(R) \times R^n \mid v \in M\}$$

We have maps $Gl_n \times \gamma_{d,n} \to \gamma_{d,n}$, $\gamma_{n,m} \to \gamma_{n,n(m+1)}$ which are short for commutative diagrams

$$\begin{array}{ccc} Gl_n \times T(d,n) & \longrightarrow & T(d,n) \\ \downarrow & & \downarrow \\ Gl_n \times Gr(d,n) & \longrightarrow & Gr(d,n) \end{array} \qquad \begin{array}{ccc} T(n,m) & \longrightarrow & T(n,m+1) \\ \downarrow & & \downarrow \\ Gr(n,m) & \longrightarrow & Gr(n,m+1) \end{array}$$

$R^{nm} \to R^{n(m+1)}$ n copies of $R^m \to R^{m+1}$, $v \mapsto (v,0)$. There is a map $Gl_n \times \gamma_{n,nm} \to \gamma_{n,nm}$ which come from $Gl_n \to Gl_{nm}$. We also have

$$\begin{array}{ccc} Gl_n \times \gamma_{n,nm} & \longrightarrow & \gamma_{n,nm} \\ \downarrow & & \downarrow \\ Gl_n \times \gamma_{n,n(m+1)} & \longrightarrow & \gamma_{n,n(m+1)} \end{array}$$

and this square commutes. These all restrict to actions of the symmetric groups Σ_n via an embedding $\Sigma_n \to Gl_n$. So we get maps $\gamma_{n,nm} \times \gamma_{p,pm} \to \gamma_{n+p,(n+p)m}$ which is $\Sigma_n \times \Sigma_p$-equivariant and

$$\begin{array}{ccc} \gamma_{n,nm} \times \gamma_{p,pm} & \longrightarrow & \gamma_{n,n(m+1)} \times \gamma_{p,p(m+1)} \\ \downarrow & & \downarrow \\ \gamma_{n+p,(n+p)m} & \longrightarrow & \gamma_{n+p,(n+p)(m+1)} \end{array}$$

here, $\Sigma_n \times \Sigma_p \subset \Sigma_{n+p}$ is the standard inclusion. We claim there exists a map $\chi_{n,p}$ which makes a commutative diagram

$$\begin{array}{ccc} \gamma_{n,nm} \times \gamma_{p,pm} & \longrightarrow & \gamma_{p,pm} \times \gamma_{n,nm} \\ \downarrow & & \downarrow \\ \gamma_{n+p,(n+p)m} & \xrightarrow[\chi_{n,p}]{} & \gamma_{n+p,(n+p)m} \end{array}$$

$$\begin{array}{ccc} T(n,n) & \longrightarrow & \mathbb{A}^n_S \\ \downarrow & & \downarrow \\ Gr(n,n) & \longrightarrow & S \end{array}$$

Definition. We define

$$MGL_n = colim_{m \geq n} Th(\gamma_{n,nm})$$

with the induced Σ_n-action. We have a $\Sigma_n \times \Sigma_p$ equivariant map $MGL_n \wedge MGL_p \to MGL_{n+p}$ obtained from

$$\begin{array}{ccc} Th(\gamma_{n,nm}) \wedge Th(\gamma_{p,pm}) & \longrightarrow & Th(\gamma_{n+p,(n+p)m}) \\ \downarrow & \nearrow & \\ Th(\gamma_{n,m} \times \gamma_{p,m}) & & \end{array}$$

Σ_n equivariant maps $\iota_n : T^{\wedge n} \to MGL_n$ $n \geq 0$.

If $n = 0$ then $S \to Th(id_S)$. If $n \geq 1$ $T^{\wedge n} = Th(\gamma_{1,1}) \to Th(\gamma_{1,1} \wedge \cdots \wedge \gamma_{1,1}) = Th(\gamma_{n,n}) \to MGL_n$

Then

$$MGL = \{MGL_n, T \wedge MGL_n \to MGL_{n+1}\}_{n \geq 0}$$

is a motivic symmetric ring spectra which is commutative.

Remark 15. Not only do the symmetric groups act, but this spectrum has a natural action of the general linear groups.

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