Motives and Milnor conjecture

notes by S. Kelly:

Day IV

Contents

1 A. Østvær/ O. Röndigs, Homotopy theory of schemes IV

The plan is to discuss the following three topics:

1. Motivic cohomology in $SH(k)$.
2. Thom isomorphism.
3. The degree map.

1.1 Motivic cohomology

The field k will always be perfect. Recall that we have an adjunction

$$\mathbb{Z}_{tr} : MS_\bullet \leftrightarrows AddFun(SmCorr(k)^{op}, sAb) : \mathcal{U}$$

and that there is a canonical isomorphism $\mathbb{Z}_{tr}(\mathcal{X} \wedge \mathcal{Y}) \cong \mathbb{Z}_{tr}(\mathcal{X}) \otimes_{tr} \mathbb{Z}_{tr}(\mathcal{Y})$.

Definition. The spectrum that represents motivic cohomology has as dth term $\mathsf{H}\mathbb{Z}_d = \mathcal{U}\mathbb{Z}_{tr}(T^d)$ where $T^d = (T)^{\wedge d}$. The bonding maps come from the unit of the adjunction $T^d \to \mathcal{U}\mathbb{Z}_{tr}(T^d) = \mathsf{H}\mathbb{Z}_d$ and the multiplication

$$\mathsf{H}\mathbb{Z}_d \wedge \mathsf{H}\mathbb{Z}_e = \mathcal{U}\mathbb{Z}_{tr}(T^d) \wedge \mathcal{U}\mathbb{Z}_{tr}(T^e) \stackrel{\mu_{d,e}}{\to} \mathcal{U}\mathbb{Z}_{tr}(T^{d+e}) = \mathsf{H}\mathbb{Z}_{d+e}.$$

Explicitely, they are defined as the composition

$$T \wedge \mathsf{H}\mathbb{Z}_d \to \mathsf{H}\mathbb{Z}_1 \wedge \mathsf{H}\mathbb{Z}_d \to \mathsf{H}\mathbb{Z}_{1+d}.$$

Lemma 1. *This is a motivic spectrum. In fact, a commutative motivic symmetric ring spectrum. This works for any commutative ring R.*

Recall: if $\mathcal{X}$ is a pointed motivic space, we define $(Sing^{\mathbb{A}^1}\mathcal{X}(Z))_n = \mathcal{X}(Z \times \Delta^n)_n$. We have seen that $Sing^{\mathbb{A}^1}\mathsf{H}\mathbb{Z}_d = \mathcal{U}C^{\mathbb{A}^1}\mathbb{Z}_{tr}(T^d) \cong \mathcal{U}C^{\mathbb{A}^1}\mathbb{Z}_{tr}(\mathbb{P}^d/\mathbb{P}^{d-1})$. The term on the left is fibrant in the level-wise model structure. By construction the map $E \to Sing^{\mathbb{A}^1}E$ for any object E is an $\mathbb{A}^1$-weak equivalence. We then consider the morphism $\mathsf{H}\mathbb{Z}_d \to \underline{hom}(T, Sing^{\mathbb{A}^1}\mathsf{H}\mathbb{Z}_{d+1}) = \mathcal{U}\underline{hom}(\mathbb{Z}_{tr}(T), C^{\mathbb{A}^1}\mathbb{Z}_{tr}(T^{d+1}))$. The cancellation theorem gives us $\mathbb{Z}(d)[-1] \cong \mathbb{Z}(d+1)_{-1}$. So $Sing^{\mathbb{A}^1}\mathsf{H}\mathbb{Z}$ is a fibrant motivic spectrum.

For $X \in Sm/k$ we have

$$\begin{aligned} \mathsf{H}\mathbb{Z}^{p,q}(X) &= \hom_{SH(k)}(\Sigma_T^\infty X_+, S^{p,q} \wedge \mathsf{H}\mathbb{Z}) \\ &= \hom_{SH(k)}(\Sigma_T^\infty X_+, \mathcal{U}\mathbb{Z}_{tr}(S^{p,q}) \\ &= H_M^{p,q}(X, \mathbb{Z}) \end{aligned}$$

Now because $\mathsf{H}\mathbb{Z}$ is a ring spectrum, the terms on the left form a graded ring, and in fact, the stated isomorphisms are isomorphisms of graded rings. The canonical morphism $T^d \to \mathsf{H}\mathbb{Z}_d$ corresponds to an element $t \in \mathsf{H}\mathbb{Z}^{2d,d}(T^d)$ the tautological class and smashing with t is an isomorphism

$$\mathsf{H}\mathbb{Z}^{p,q}(E) \xrightarrow{\sim} \mathsf{H}\mathbb{Z}^{p+2d,q+d}(T^d \wedge E).$$

Despite the extra subtlety of the construction via spectra, the motivic stable homotopy category is actually simpler than the unstable motivic homotopy category. Why not just stick to DM^{eff}? Because its not complicated enough! We need the object $H\mathbb{F}_2^{*,*}(H\mathbb{F}_2)$ or at least, some elements of it. In particular, the following.

$$\mathsf{H}\mathbb{Z} \xrightarrow{2} \mathsf{H}\mathbb{Z} \xrightarrow{p} H\mathbb{F}_2 \xrightarrow{\delta} S^{1,0} \wedge \mathsf{H}\mathbb{Z}$$

We define the Steenrod square as $Sq^1 = S^{1,0} \wedge p \circ \delta \in H\mathbb{F}_2^{1,0}(H\mathbb{F}_2)$.

1.2 The Thom isomorphism

We will discuss an identification $H^{\bullet,\bullet}(X_+)$ with $H^{\bullet,\bullet}(Th_X(V))$ for a vector bundles $V \to X$.

Definition. Let $V \to X$ be a vector bundle of rank $r+1$ and $X \in Sm/k$. Over the corresponding projective bundle $\mathbb{P}(V)$ there is the tautological bundle $\mathcal{O}(1)$ and this gives a class $c_V \in H^{2,1}(\mathbb{P}(V)_+) \cong Pic(\mathbb{P}(V))$. We define

$$\begin{aligned} \phi_V : \oplus_{i=0}^r H^{\bullet,\bullet}(X_+) &\to H^{\bullet,\bullet}(\mathbb{P}(V)) \\ (a_0, \dots, a_r) &\mapsto \sum_{i=0}^r p^*(a_i) \cdot c_V^i \end{aligned}$$

Remark 2.

1. In the case of a trivial bundle of rank 2 this c corresponds to $c_{\mathcal{O}^2} = -t \in H^{2,1}(\mathbb{P}^1) \cong H^{2,1}(T))$.

2. The projective bundle theorem says that ϕ_V is an isomorphism.

Consider the Thom diagonal $Th_X(V) \to X_+ \wedge Th_X(V)$.

Corollary 3 (Thom isomorphism)**.** *Let $V \to X$ be a vector bundle of rank r. Then there exists a unique class $t_V \in H^{2r,r}(Th_X V)$ such that*

1. *the map $H^{*,*}(X_+) \to H^{*+2r,*+r}(Th_X V)$ sending a to $a \wedge t_V$ is an isomorphism, and*
2. *t_V restricts to the tautological class in $H^{2r,r}(Th_\eta V)$ for every generic point η of X.*

Proof.

$$\begin{array}{ccccc} V - z(X) & \longrightarrow & V & \longrightarrow & Th_X(V) \cong \mathbb{P}(V \oplus \mathcal{O})/\mathbb{P}(V) \\ \downarrow & & \downarrow & & \\ \mathbb{P}(V) \cong \mathbb{P}(V \oplus \mathcal{O}) - \mathbb{P}(\mathcal{O}) & \longrightarrow & \mathbb{P}(V \oplus \mathcal{O}) & & \end{array}$$

$t_V = (-c_V)^r +$ something. □

Remark 4. If we take $Th_{\mathbb{P}^1}(\mathcal{O})$ this is not the same as $Th_{\mathbb{P}^1}(\mathcal{O}(1))$ (whereas in DM^{eff} every Thom space is isomorphic to the Thom space of a trivial bundle. To see that these Thom spaces are not the same, note that $Th_{\mathbb{P}^1}(\mathcal{O}) \cong T \wedge \mathbb{P}^1$ and $Th_{\mathbb{P}^1}(\mathcal{O}(1)) \cong \mathbb{P}^2$ and consider the motivic cohomology. If we take the Steenrod squares in these two graded rings, in the first case we have the zero endomorphism, but in the second case it is nonzero.

1.3 The degree map

Take $X \in PSm_k$ of dimension d. We have $H^{2d,d}(X_+) = CH^d(X) = CH_0(X)$ and we have the pushforward via the projection to a point sending this group to $CH_0(Spec\ k) = \mathbb{Z}$.

The Thom perspective: Take $i : Z \subset X$ of codimension c in Sm_k. We take the Thom space of the normal bundle and the Thom class in its cohomology $t_{N(i)} \in H^{2c,c}(Th_Z N(i))$ via the homotopy purity isomorphism $hp : Th_Z N(i) \to X/X - Z$ this is isomorphic to $H^{2c,c}(X/X - Z)$ and the Thom class is sent to some $a_{X,Z} = a_Z \in H^{2c,c}(X/X - Z)$.

Lemma 5. *Suppose we have $Z \subset Y \subset X$ a chain of closed immersions (for example, a diagonal of projective spaces followed by a Segre embedding). We have a map*

$$f : X/X - Y \to X/X - Z.$$

Then $f^*(a_{X,Z}) = hp^* t_{N(Y \subset X)} \wedge a_{Y,Z}$.

Proof. Homotopy purity. □

Now suppose that k is algebraically closed and consider $a \in H^{2d,d}(X_+)$ where $\dim X = d$. Since $\mathbb{Z}(d)$ has first nonvanishing homology in d there exists some closed subscheme $Z \subset X$ of codimension d such that $a|_{X-Z} = 0$. So $a \in H^{2d,d}(X/X - Z)$. This shows that under thes conditions $H^{2d,d}(X_+)$ is generated by the elements a_x where x is a rational point of X.

Definition. The degree is the unique natural transformation

$$\{H^{2\dim X, \dim X}(X_+) \to \mathbb{Z}\}_{k \subset k', X \in PSm_{k'}}$$

such that

1. it is compatible with field extension, and
2. if $x \in X(k)$ the $\deg a_x = 1$.

Theorem 6 (Voevodsky)**.** *Let X be a smooth projective k-variety of dimension d. There exists a natural number $n = n(X) \in \mathbb{N}$ depending on X, and a vector bundle $V \to X$ such that*

1. *$[Tan_X] + [V] = [\mathcal{O}^{d+n}] \in K_0(X)$ (the former is the class of the tangent bundle),*
2. *there exists a map $T^{d+n} \to Th_X(V)$ such that*

   $$H^{2d,d}(X_+) \stackrel{Thom}{\to} H^{2d+n,d+n}(Th_X(V)) \stackrel{\rho_V^*}{\to} H^{2(d+n),d+n}(T^{d+n}) \stackrel{Thom}{\to} H^{0,0}(k) \cong \mathbb{Z}.$$

   is the degree map.

Proof. We consider first the case $X = \mathbb{P}^d$ and the general case will then follow quite easily.

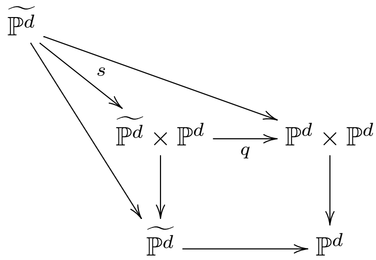
We have a short exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O}^{d+1} \to Tan(-1) \to 0$ from which we obtain $0 \to Cot \to \mathcal{O}(-1)^{d+1} \to \mathcal{O} \to 0$ by taking the dual and then twisting. It follows that $[Tan(-1)] = (d+1)[\mathcal{O}] - [\mathcal{O}(-1)]$ and $[Cot] + [\mathcal{O}] = (d+1)[\mathcal{O}(-1)]$ in $K_0(\mathbb{P}^d)$. Define $W = Cot \oplus (Cot \otimes Tan) \to \mathbb{P}^d$.

$$\begin{aligned}
[Tan] + [W] &= [Cot] + [Cot][Tan] + [Tan] \\
&= ([Cot] + [\mathcal{O}])[Tan + [Cot] \\
&= (d+1)[Tan(-1)] + [Cot] \\
&= (d+1)\Big((d+1)[\mathcal{O}] - [\mathcal{O}(-1)]\Big) + [Cot] \\
&= (d^2 + 2d)[\mathcal{O}]
\end{aligned}$$

and so W satisfies the requirements. This gives a stable map but we want an unstable morphism. We use Jouanolou's trick to reduce to an affine variety. Define

$$H = \{(x, y) \in \mathbb{P}^d \times \mathbb{P}^d : \sum x_i y_i = 0\} \subset \mathbb{P}^d \times \mathbb{P}^d$$

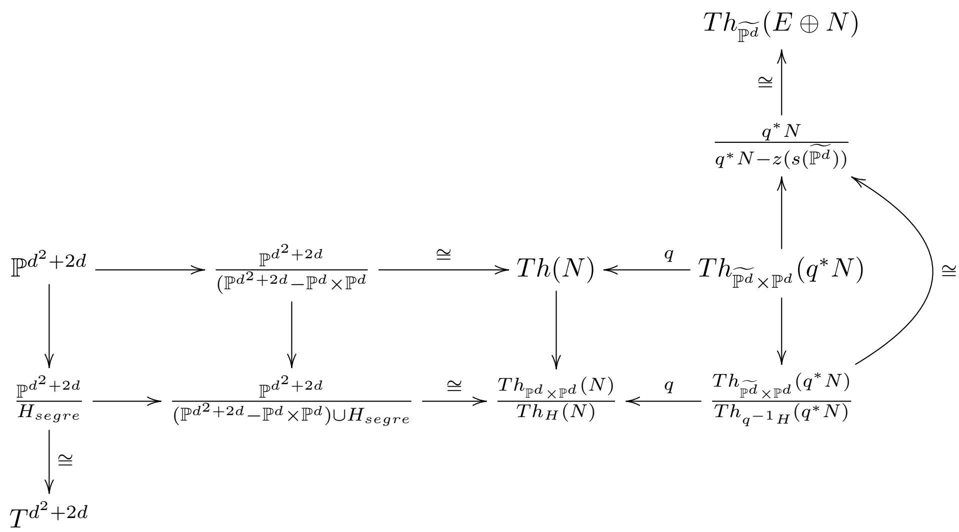
and take $p : \widetilde{\mathbb{P}^d} \subset \mathbb{P}^d \times \mathbb{P}^d - H \to \mathbb{P}^d$. This is an $\mathbb{A}^d$-fiber bundle, and therefore a motivic weak equivalence. Moreover, $\widetilde{\mathbb{P}^d}$ is affine: we take the Segre embedding $\mathbb{P}^d \times \mathbb{P}^d \to \mathbb{P}^{d^2+2d}$ and H is sent to the intersection with a hyperplane H_{segre}, hence H is a closed subscheme of an affine space.



We denote $E = E(s)$ the normal bundle of this section s and $N = N(Segre)$ the noral bundle to the Segre embedding. We claim that there is an $m = m(d) \in \mathbb{N}$ such that $E \oplus N \oplus \mathcal{O}^m = p^*W \oplus \mathcal{O}^m$. Since $\widetilde{\mathbb{P}^d}$ is affine, it suffices to prove that $[E] + [N] = p^*[W] \in K_0(\widetilde{\mathbb{P}^d})$. One has $E \cong pr_2^* Tan_{\mathbb{P}^d}$ and there is a short exact sequence

$$0 \to Tan_{\widetilde{\mathbb{P}^d}} \to \underbrace{Tan_{\mathbb{P}^{d^2+d}}|_{\widetilde{\mathbb{P}^d}}}_{\cong \mathcal{O}^{d^2+2d}} \to N \to 0.$$

This gives a map $T^m \wedge Th_{\widetilde{\mathbb{P}^d}}(E \oplus N) = Th_{\widetilde{\mathbb{P}^d}}(E \oplus N \oplus \mathcal{O}^m) \cong Th_{\widetilde{\mathbb{P}^d}}(p^*W \oplus \mathcal{O}^m) \to Th_{\mathbb{P}^d}(W \oplus \mathcal{O}^m) = T^m \wedge Th_{\mathbb{P}^d}(W)$.



It is easy to see that $q^{-1}H \to \widetilde{\mathbb{P}^d} \times \mathbb{P}^d - s(\widetilde{\mathbb{P}^d})$ is a motivic weak equivalence. This is a section of an $\mathbb{A}^1$-bundle $\sum x_i y_i \neq 0, y \neq z$, $(x, y, z) \mapsto (x, y, x^\perp)$.

Consequently $Th_{q^{-1}H}(q^*N) \cong Th_{\widetilde{\mathbb{P}^d} \times \mathbb{P}^d - s(\widetilde{\mathbb{P}^d})}$.

Recall the map $f : T^{d^2+2d+m} \to Th_{\widetilde{\mathbb{P}^d}}(E \oplus N \oplus \mathcal{O}^m) \to Th_{\mathbb{P}^d}(W \oplus \mathcal{O}^m)$. We claim that the induced map sends a_* to 1 for every $x \in \mathbb{P}^d(k)$.

$$H^{2d,d}(\mathbb{P}^d_x) \cong H^{2(d+r),d+r}(Th_{\mathbb{P}^d}(W \oplus \mathcal{O}^m)) \xrightarrow{f^*} H^{2(d^2+2d+m),d^2+2d+m}(T^{\cdot\ \cdot}) \cong H^{0,0}(k)$$

Using the lemma about Thom spaces of sequences of closed embeddings, we see that $a_x \mapsto a_{Segre(x,x)} \sim$ Tautological class.

Now for a smooth closed subscheme $i : X \subset \mathbb{P}^D$ of $\mathbb{P}^D$ we have $W \oplus \mathcal{O}^m$ over $\mathbb{P}^D$ and we define $V = N(X \to \mathbb{P}^D \xrightarrow{z} W \oplus \mathcal{O}^m)$.

$$T^{m+D^2+2D} \to Th_{\mathbb{P}^D}(W \oplus \mathcal{O}^m) \xrightarrow{pr} W \oplus \mathcal{O}^m/(W \oplus \mathcal{O}^m) - Z - (X) \xrightarrow{\cong} Th_Z(V)$$

So we can identify the normal bundle as $V = i^*(W \oplus \mathcal{O}^m) \oplus N(i)$. We have firstly $[Tan_X] + [V] = [\mathcal{O}^r]$ in $K_0(X)$ and secondly we know that [Voevodsky 2.4] a_x corresponds to $a_{i(x)} \in H^{2d,d}(\mathbb{P}^d) \cong H^{2d+,d+}Th(W \oplus \mathcal{O}^m)$. □

1.4 Bonus section

Recall the K-theory spectra $\mathcal{K} : Sm^{op} \to sSet_\bullet$ which takes X to $\Omega(NQVect(X))^f$. This has another description where we take $\Omega(BN(Vect(X), iso))^f$ where B is the group completion. The obvious morphism between these two is a Nisnevich local weak equivalence. We can also take $\Omega(B \amalg_{n\in\mathbb{N}} BGL_n(X))^f$ (i.e. taking the category of trivial bundles with isomorphisms). Morel-Voevodsky show that $BGL_\infty \times \mathbb{Z}$ is weakly equivalent to this latter where $BGL_\infty = colim BGL_n$. Up to $\mathbb{A}^1$ weak equivlance, BGL_n can be identified with the Grassmanian $Gr_n(\mathcal{O}^\infty) = colim Gr_n(\mathcal{O}^{m+n})$. For example, for $n = 1$ we have $GL_n = \mathbb{G}_m$ and $B\mathbb{G}_m \cong \mathbb{P}^\infty = \mathbb{A}^\infty - 0/\mathbb{G}_m$.