Motives and Milnor conjecture

notes by S. Kelly:

Day V

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1.1 Conjectures, equivalent formulations and complements.

Let F be a field and ℓ a prime different from the characteristic of F. Recall that we have the Norm residue homomorphism

$$K_n^M(F)/\ell \to H^n_{et}(Spec\ F, \mu_\ell^{\otimes n}) \qquad \text{(Norm residue map)}$$

(where étale cohomology is identified with Galois cohomology in the usual way).

Bloch-Kato Conjecture.

$(BK(\ell, n))$ *The norm residue homomorphism is an isomorphism in degree n.*

Weak Block-Kato Conjecture.

$(BK'(\ell, n))$ *The norm residue homomorphism is surjective in degree n.*

We reinturpret both sides of the norm residue map. LHS: In Joseph's lecture we saw that $H^{n,n}(F, \mathbb{Z}) = \mathbb{H}^n_{Nis}(F, \mathbb{Z}(n)) = K_n^M(F)$ where $\mathbb{Z}(1) = \mathcal{O}^*[-1]$. Associated to $\mathbb{Z} \xrightarrow{\ell} \mathbb{Z} \to \mathbb{Z}/\ell$ we have a long exact sequence

$$H^{n,n}(Spec\, F, \mathbb{Z}) \xrightarrow{\ell} H^{n,n}(Spec\, F, \mathbb{Z}) \to H^{n,n}(Spec\, F, \mathbb{Z}/\ell) \to H^{n+1,n}_{Nis}(Spec\, F, \mathbb{Z}) = 0$$

(where the last is zero because $H^i_{Nis}(\mathbb{Z}(n)) = 0$ if $i > n$) and hence an isomorphism

$$H^{n,n}(Spec\ F, \mathbb{Z}/\ell) \cong K_n^M(F)/\ell.$$

RHS: Sheafifying $\mathcal{O}^* \xrightarrow{\ell} \mathcal{O}^*$ for the étale topology we find $\mathbb{Z}/\ell(1)_\ell = \mu_\ell$. One can show that $\mu_\ell^{\otimes q} \cong (\mathbb{Z}/\ell(1)_{et})^{\otimes q}$. This gives us an identification $\mathbb{Z}/\ell(q)_{et} \cong$

$\mu_\ell^{\otimes q}$. As a consequence $\mathbb{H}^i_{et}(F, \mathbb{Z}/\ell(q)_{et}) = H^i_{et}(F, \mu_\ell^{\otimes q}$. So we have reinturprettations of both sides, and now we look for a map between the two which correpsonds to the Norm residue map.

There is a morphism of sites

$$\alpha : (Sm_k)_{et} \to (Sm)_{Nis}$$

which induces a pair of adjoint functors

$$R\alpha_* : D^- Shv_{et} SmCor_k \leftrightarrows D^- Shv_{Nis} SmCor_k : a^*$$

For any coefficient group A we get $A(q)$ (A is an abelian group) and we get a map $A(q) \to Ra_* a^* A(q)$ and this map factors through an appropriate truncation $A(q) \to \tau^{\leq q} Ra_* a^* A(q)$.

Definition. We define

$$B_\ell(n) = \tau^{\leq n} Ra_* a^* \mathbb{Z}/\ell(q)$$
$$L(n) = \tau^{\leq n} Ra_* a^* \mathbb{Z}(q)$$
$$K(n) = Cone(\mathbb{Z}(n) \to L(n))$$

With these definitions, $(BK(\ell, n))$ is equivalent to:

Beilinson-Lichtenbaum Conjecture.

$(BL(\ell, n))$ The map $\alpha_n : \mathbb{Z}/\ell(n) \to B_\ell(n)$ is a quasi-isomorphism.

Lemma 1. *If X is a smooth scheme and $i \leq n$ then $\mathbb{H}^n_{Nis}(X, B_\ell(n)) \tilde{\to} H^i_{et}(X, \mu_\ell^{\otimes n})$ is an isomorphism.*

Recall the following results of Voevodsky that we have seen.

1. If F is a homotopy invariant presheaf with transfers then the cohomology presheaves are again homotopy invariant.
2. The Nisnevich sheafification of a homotopy invariant presheaf with transfers is a homotopy invariant Nisnevich sheaf with transfers.
3. If $F \to G$ is a morphism of homotopy invariant Nisnevich sheaves with transfers then f is an isomorphism if and only if the induced map $F(E) \to G(F)$ is an isomorphism for all finitely generated field extensions.

If $(BL(\ell, n))$ holds, then $H^n(\mathbb{Z}/\ell(n)) \to H^n(B_\ell(n))$ is an isomorphism and so there is an isomorphism $K^M_n(E) \to H^n_{et}(E, \mu_\ell^{\otimes n})$ for all finitely generated field extensions E/F. We have to check that the map $K^M_n(E) \to H^n_{et}(E, \mu_\ell^{\otimes n})$ induced by the above identifications is the norm-residue homomorphism.

Claim: When α_1 coincides with the map induced by Kummer theory.

Claim:

(a) There are natural pairings $B_\ell(n) \otimes B_\ell(n) \to B_\ell(n+m)$ such that for $i \leq n, j \leq m$ and any smooth scheme X, the induced pairings $\mathbb{H}^i_{et}(X, B_\ell(n)) \otimes \mathbb{H}^j_{et}(X, B_\ell(m)) \to \mathbb{H}^{i+j}_{et}(X, B_\ell(n+m))$ coincide with the corresponding pairings in étale cohomology.

(b) The diagram

$$\begin{array}{ccc} \mathbb{Z}/\ell(m) \otimes \mathbb{Z}/\ell(n) & \longrightarrow & \mathbb{Z}/\ell(m+n) \\ \downarrow & & \downarrow \\ B_\ell(m) \otimes B_\ell(n) & \longrightarrow & B_\ell(m+n) \end{array}$$

commutes.

Corollary 2 (independant of the claims)**.** *The map $\alpha_n : K_n^M(E) \to H^n_{et}(E, \mu_\ell^{\otimes n})$ is the norm residue homomorphism.*

Corollary 3 (of the above comutation)**.** *If $BL(\ell, n)$ holds then $BK(\ell, n)$ holds for al E finitely generated over F.*

Lemma 4. *If $BL(\ell, n)$ holds then $BL(\ell, n-1)$ holds as well.*

Proof. We will see the proof later. □

Conversely,

Theorem 5 (Suslin-Voevodsky, Geisser-Levine, Suslin)**.** *If $(BK'(\ell, i))$ holds for all E finitely generated over F and all $i \leq m$ then $BL(\ell, i)$ holds or all $i \leq m$.*

Remark 6.

1. Suslin-Voevodsky proved this result assuming resolution of singularities, Geisser-Levine removed the ROS hypothesis and then Suslin gave a more streamlined proof. The key in Suslins proof is the cancellation theorem.

2. We will need a bit more than this because the statement is about $\mathbb{Z}/\ell$ coefficients. There are variants of $BL(\ell, n)$ with $\mathbb{Q}/\mathbb{Z}_{(\ell)}$ and $\mathbb{Z}/\ell^m$ coefficients and these variants hold.

Theorem 7. *The following are equivalent.*

1. *$BL(\ell, n)$ holds.*

2. *The group $H^{n+1,n}_{et}(E, \mathbb{Z}_{(\ell)})$ is zero for all finitely generated E/F.*

Proof. Some lemmas.

Lemma 8. *If X is a smooth scheme, then the maps $H^{p,q}(X, \mathbb{Q}) \to H^{p,q}_{et}(X, \mathbb{Q})$ are isomorphisms.*

Proof. Uses the equivalence of $DM^{eff}(\mathbb{Q})$ and $D^{eff,et}(\mathbb{Q})$. □

Lemma 9. *The complexes $Ra_*a^*\mathbb{Z}_{(\ell)}(q)$ have homotopy invariant cohomology sheaves.*

Proof. Using the coefficient sequence $\mathbb{Z}_{(\ell)} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}_{(\ell)}$ it suffices to show $H^{p,q}_{et}(X, \mathbb{Z}_{(\ell)})$ is homotopy invariant

$$H^{p,q}_{et}(X, \mathbb{Q}/\mathbb{Z}_{(\ell)}) \to H^{p,q}_{et}(X, \mathbb{Z}_{(\ell)}) \to H^{p,q}_{et}(X, \mathbb{Q})$$

The latter is homotopy invariant by the $\mathbb{Q}$ coefficients statement, and the former by homotopy $\mathbb{Z}/\ell^n$ invariance of étale cohomology. □

(ii) implies (i): Consider again the coefficient sequence $\mathbb{Z}_{(\ell)} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}_{(\ell)}$. We get a sequence

$$\begin{array}{ccccc} H^{n,n}(E, \mathbb{Q}) & \longrightarrow & H^{n,n}(E, \mathbb{Q}/\mathbb{Z}_{(\ell)}) & \longrightarrow & H^{n+1,n}(E, \mathbb{Z}_{(\ell)}) \\ \downarrow \cong & & \downarrow & & \downarrow \\ H^{n,n}_{et}(E, \mathbb{Q}) & \longrightarrow & H^{n,n}_{et}(E, \mathbb{Q}/\mathbb{Z}_{(\ell)}) & \longrightarrow & H^{n+1,n}_{et}(E, \mathbb{Z}_{(\ell)}) \end{array}$$

The groups on the right are zero by cohomological dimension considerations, and we assume that $H^{n+1,n}_{et}(E, \mathbb{Z}_{(\ell)}) = 0$. Surjectivity of the center morphism is equal to weak BK with $\mathbb{Q}/\mathbb{Z}_{(\ell)}$ coefficients. By Geisser-Levine with $\mathbb{Q}/\mathbb{Z}_{(\ell)}$ coefficients we have BL with $\mathbb{Q}/\mathbb{Z}_{(\ell)}$ coefficients. The sequence $\mathbb{Z}/\ell \to \mathbb{Q}/\mathbb{Z}_{(\ell)} \xrightarrow{\ell} \mathbb{Q}/\mathbb{Z}_{(\ell)}$ together with a diagram chase gives the weak Bloch-Kato conjecture, which implies (i) by Geisser-Levine. □

We finish with a last conjecture.

Hilbert 90 Conjecture.

($H90(n, \ell)$) The group $H^{n+1,n}(F, \mathbb{Z}_{(\ell)}) = 0$.

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Consider the case $\ell = 2$ with the characteristic of F different from 2.

$$BK(2, n) = MC(n) \Leftarrow BL(n) \Leftrightarrow MH90(n)$$
$$h^n : \underbrace{K^M_n(F)/2}_{=k_n(F)} \xrightarrow{\sim} \underbrace{H^n_{et}(F, \mu_2^{\otimes n})}_{=H^n(F)}$$

We consider the conjecture

Motivic Hilbert 90 Conjecture.

$MH90(n)$ $H^{n+1,n}_{et}(F, \mathbb{Z}_{(2)}) = 0$.

We will use induction on n to prove Motivic Hilbert 90. Let L/F be a quadratic extension, $G = Gal(L/F) = \langle\sigma\rangle$.

$H90(n)$ $K_n^M \overset{1-\sigma}{\to} K_n^M(L) \overset{N_{L/F}}{\to} K_n^M$ is exact.

Proposition 10. *Motivic Hilbert 90 for i implies Hilbert 90 for i.*

Proof. We have exact sequences

$$0 \to \mathbb{Z} \to \mathbb{Z}[G] \overset{1-\sigma}{\to} \mathbb{Z}[G] \to \mathbb{Z} \to 0$$

$$0 \to \mathbb{Z}(i) \to \mathbb{Z}[G] \otimes \mathbb{Z}(i) \to M \to 0$$

$$0 \to M \to \mathbb{Z}[G] \otimes \mathbb{Z}(i) \to \mathbb{Z}(i) \to 0$$

and $H^*_{et}(F, \mathbb{Z}[G] \otimes \mathbb{Z}(i)) = H^*_{et}(L, \mathbb{Z}(i))$ Shapiro-Faddeev. Using the corresponding long exact sequences we obtain

$$\begin{array}{ccccc} H^{i,i}_{et}(L,\mathbb{Z}_{(2)}) & \longrightarrow & H^i_{et}(F, M_{(2)}) & \longrightarrow & H^{i+1,i}_{et}(F,\mathbb{Z}_{(2)}) \\ & \overset{1-\sigma}{\searrow} & \downarrow & & \\ & & H^{i,i}_{et}(L,\mathbb{Z}_{(2)}) & & \\ & & \downarrow N & & \\ & & H^{i,i}_{et}(F,\mathbb{Z}_{(2)} & & \end{array}$$

The group on the right is zero by the assumption of $MH90(i)$ (which implies $BL(i)$) and so the diagonal complex on the left is exact.

$$H^{i,i}_{et}(*,\mathbb{Z}_{(2)}) = H^{i,i}(*,\mathbb{Z}_{(2)}) = K_i^M(*)_{(2)}$$

$$K_i^M(L)_{(2)} \overset{1-\sigma}{\to} K_i^M(L)_{(2)} \overset{N}{\to} K_i^M(F)(2)$$

□

Definition. A field is 2-special if F has no nontrivial odd degree extensions.

Some properties of 2-special fields are the following.

1. If F is 2-special, and L/F is finite, then there exists a filtration

$$F = F_0 \subset F_1 \subset \cdots \subset F_n = L$$

with F_{i+1}/F_i quadratic.

2. For every field F there exists F'/F such that F' is 2-special and for every intermediate extension $F'/L/F$ of finite degree over F, the degree is odd. (It suffices to take the field of invariant elements of an algebraic closure under a Sylow 2-subgroup H of the absolute Galois group).

Step 1: $MH90(n)$ for fields F such that

1. F is 2-special,
2. $k_n(F) = 0$ i.e. $K_n^M(F)$ is 2-divisible.

Claim 1: For any quadratic field extension L/F the norm map $N_{L/F} : K_{n-1}^M(L) \to K_{n-1}^M(F)$ is surjective.

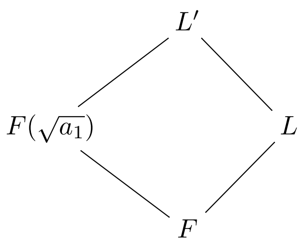
$$\begin{array}{ccccccc} k_{n-1}(L) & \xrightarrow{N} & k_{n-1}(F) & \longrightarrow & k_n(F) & & = 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ H^{n-1}(L) & \xrightarrow{N} & H^{i-1}(F) & \xrightarrow{d} & H^i(F) & & \end{array}$$

where $L = F(\sqrt{d})$. The bottom sequence is exact (a consequence of Shapiro-Faddeev, $0 \to A \to A[G] \to A \to 0$ and $H^*(F, A[G]) = H^*(L, A)$ and so N in the top row is surjective.

Claim 2: $H90(n)$ holds for any quadratic L/F. Is

$$K_n^M(L) \overset{1-\sigma}{\to} K_n^M(L) \overset{N}{\to} K_n^M(F)$$

exact? We will try to construct a homomorphism from $K_n^M(F)$ to $coker(K_n^M(L) \overset{1-\sigma}{\to} K_n^M(L))$. Take $\{a_1, \dots, a_n\}$. We can find $u \in K_{n-1}^M(L)$ such that $N_{L/F}(u) = \{a_1, \dots, a_n\}$ and we send $\{a_1, \dots, a_n\}$ to $u \cdot \{a_n\} + Im(1-\sigma)$. This does not depend on the choice of u: if $N(u) = N(u')$ then $N(u - u') = 0$ then $u \cdot \{a_n\} = u' \cdot \{a_n\}$ by $MH90(n-1) \implies H90(n-1) \implies u - u' \in Im(1-\sigma)$.



We take $u \in L$ and we have

$$N(u_{L'} - \{\sqrt{a_1}, a_2, \dots, a_{n-1}\} = N(u)_{F'} - \{a_1, a_2, \dots, a_{n-1}\} = 0$$

$H90(n-1) \implies u_{L'} - \{\sqrt{a_1}, \dots, a_{n-1}\} = (1-\sigma)w$ for $w \in K_{n-1}^M(L')$.

$u \cdot \{a_n\} = u \cdot (N_{L'/L}(\{1-\sqrt{a_1}\})) = N_{L'/L}(u_{L'} \cdot \{1-\sqrt{a_1}\}) = N_{L'/L}((1-\sigma)w + \{\sqrt{a_1}, \dots, a_{n-1}\{1-\sqrt{a_1}\}) = N_L$

Claim 3:

$$k_n(F) \to k_n(L) \to k_n(F)$$

is exact.

We take $u \in K_n^M(L), N(u) = 2v$, $v \in K_n^M(F)$. $N(u - v_L) = N(u) - N(v_L) = N(u) - 2u = 0$.

Claim 2 then implies that $u - v_L = (1-\sigma)w$ for $w \in K_n^M(L)$, $u = v_L + (1+\sigma)w = 2w + v_L - (1+\sigma)w = 2w + (v + N(w))$.

Claim 4: $k_n(E) = 0$ for any finite field extension E/F. Recall that we have assumed that $k_n(F) = 0$. It follows from Claim 3 then that $k_n(E) = 0$ for any quadratic extension, and then by the assumption that F is 2-special and the filtration we see the claim by induction.

Claim 5. $H^n(F) = 0$. Pick an element $u \in H^n(F)$. Every element has a finite splitting field (i.e. a field extension E where $u_E = 0$), and we use induction on the degree of this splitting field. We can find an intermediate extension $E/K/F$ where $[E : F] = 2$.

$$\begin{array}{ccccc} k_{n-1}^M & \longrightarrow & k_n^M(K) & & \\ \cong\downarrow & & \downarrow & & \\ H^{n-1}(K) & \longrightarrow & H^n(K) & \longrightarrow & H^n(E) \end{array}$$

THe bottom row is exact, and $k_n(K)$ is zero. Since $u_E = 0$ it follows from a diagram chase that $u = 0$.

Claim 6: $H_{et}^{n+1,n}(F, \mathbb{Z}_{(2)}) = 0$. We consider the short exact sequence

$$0 \to \mathbb{Z}_{(2)} \xrightarrow{2} \mathbb{Z}_{(2)} \to \mathbb{Z}/2 \to 0$$

This gives an exact sequence

$$H^n(F) \to H_{et}^{n+1,n}(F, \mathbb{Z}_{(2)}) \xrightarrow{2} H_{et}^{n+1,n}(F, \mathbb{Z}_{(2)})$$

the first group is zero by claim 5 so it suffices to show that $H_{et}^{n+1,n}(F, \mathbb{Z}_{(2)})$ is 2-primary torsion. This follows from

$$H_{et}^{n+1,n}(F, \mathbb{Z}_{(2)}) \otimes \mathbb{Q} = H_{et}^{n+1,n}(F, \mathbb{Q}) = H^{n+1,n}(F, \mathbb{Q}) = 0.$$

This completes the first step of the proof.

Now for a given field F we construct a field extension $\tilde{F}/F$ such that

1. $\tilde{F}$ is 2-special,
2. $k_n(\tilde{F}) = 0$.
3. $H_{et}^{n+1,n}(F, \mathbb{Z}_{(2)}) \to H_{et}^{n+1,n}(\tilde{F}, \mathbb{Z}_{(2)})$ is injective.

Step 1 and the first two conditions implie that $MH90(n)$ holds for $\tilde{F}$. This together with the last conditions implies $MH90(n)$ for F.

Take $\alpha = \{a_1, \ldots, a_n\} \in K_n(F)$. Now α is divisible by 2 if and only if $\langle\langle a_1, \ldots, a_n \rangle\rangle = 0$ (the correpsonding Pfister form is zero) if and only if $\langle\langle a_1, \ldots, a_{n-1} \rangle\rangle \perp \langle -a_n \rangle$ is isotropic. Let X_α be the quadratic hypersurface given by $\langle\langle a_1, \ldots, a_{n-1} \rangle\rangle \perp \langle -a_n \rangle$. Suppose that $H_{et}^{n+1,n}(F, \mathbb{Z}_{(2)}) \to$

$H^{n+1,n}_{et}(F(X_\alpha(,\mathbb{Z}_{(2)})$ is injective. α is divisible by 2 in $K^M_n(F(X_\alpha))$ so we have killed one symbol.

For any finite set of symbols S in $K_n(F)$ we construct $X_S = \prod_{\alpha\in S} X_\alpha$. We can take the colimit over all finite sets of symbols $F' = colim F(X_S)$ and every symbol in $K^M_n(F)$ is divisible by 2 in $K^M_n(F')$ but we might have introduced new symbols that are nonzero, and it may not by 2-special.

So we take $F \subset F' \subset F'' \subset F''' \subset \ldots$ where F'' is the 2-special closure of F''. Then we kill all the quadrics again to get F''', etc. Finally, we take the union $\tilde{F}$ over all these fields. So the only statement left to prove is: $H^{n+1,n}_{et}(F,\mathbb{Z}_{(2)}) \to H^{n+1,n}_{et}(F(X_\alpha),\mathbb{Z}_{(2)})$ is injective.