Motives and Milnor conjecture

notes by S. Kelly:

Day VI

Contents

1 S. Gille, Chow motives and motives of quadrics

1. Rost motive. [Ros98] 1998 (this is the Latex-date but the work comes from 1987 at the same time as Rust's (not Rost) landing in Moscow).
2. Exact triangle of Voevodsky/Orlov.

1.1 Karpenko's construction of the Rost motive.

Definition. Let X_q be the projective quadric of the qadratic form q. Then a projector $\rho \in CH_{\dim\ X_q}(X_q \times X_q) = End(X_q)$ is called a Rost projector (or idempotent) if $\rho_{\overline{F}} = [\overline{X}_q \times pt] + [pt \times \overline{X}_q]$.

Example 1. If X_q is isotropic then $[X_q \times pt] + [X_q \times pt]$ is a Rost projector. Using the Rost decomposition it is easy to see that there is exactly one Rost projector.

Theorem 2 (Rost)**.** *Consider $\alpha = \{\alpha_1, \ldots, \alpha_n\}$, let $a_\alpha = \langle\langle a_1, \ldots, a_{n-1}\rangle\rangle \perp \langle -a_n\rangle$ and $X_\alpha = X_{a_\alpha}$. Then X_α posses a Rost projector p_α. The motive $M_\alpha = (X_\alpha, p_\alpha)$ is called the Rost motive of α.*

Remark 3.

1. p_α is the unique projector such that $p_\alpha \times \overline{F} \cong [\overline{X}_\alpha \times p] + [pt \times \overline{X}_\alpha]$.
2. Rost calls a projector $p \in CH_{\dim X_\alpha}(X_\alpha \times X_\alpha)$ a Rost projector if $(X_\alpha, id_{X_\alpha}, -p_\alpha) \cong Z(1)$ where Z is a projective quadric of dimension $\dim X_\alpha - Z$ (this is equivalent to our definition of a Rost projector for X_α).

Proof. The proof uses Rost nilpotence, i.e. we show that $[\overline{X}_q \times pt] + [pt \times \overline{X}_q] \in im\ (CH_{\dim\ X_q}(X_q \times X_q) \to CH_{\dim\ X_q}(\overline{X}_q \times \overline{X}_q))$. To do this we recall the definition of a split quadric form. $CH_*(X_q)$

$$\dim X_1 = \begin{cases} 2m+1 \\ 2m \end{cases}$$

i.e. q has a totally isotropic subspace V of dimension $m+1$.

$$\begin{array}{ccc} X_q & \longrightarrow & \mathbb{P}_F^{\dim X_q} \\ & \nwarrow & \uparrow \\ & & \mathbb{P}(V) \qquad \supset \mathbb{P}(V_{m-n}) \supset \cdots \supset pt \end{array}$$

Classes c_i coming from the $\mathbb{P}(V_i)$ well defined by $\mathbb{P}(V_i) \subseteq \mathbb{P}(V) \subset X_q$ in $CH_n(X_q)$.

Classical fact: Let $h \in CH^1(X_q)$ be the class of a hyperplane section. Then $c_0, \ldots, c_m, h^d, h^{d-1}, \ldots, h, [X_q]$ are a basis of the free $\mathbb{Z}$-module $CH_*(X_q)$. We have the relation $hc_i = c_{i-1}$. If X_q is split then the motive of $X_{\overline{q}}$ is split which implies that $CH(X_q) \otimes CH(X_q) \xrightarrow{\sim} CH(X_q \times X_q)$ is an isomorphism.

We start the proof with a lemma.

Let $q_\alpha = \langle\langle a_1, \ldots, a_{n-1}\rangle\rangle \perp \langle -a_n \rangle \subseteq \pi = \langle\langle a_1, \ldots, a_n\rangle\rangle$ and consider the embedding $i : X_\alpha \to X_\pi$.

Lemma 4. *Let* $\dim X_\pi = 2s$. *Then the cycle* $[\overline{X}_\pi \times c_s] + [c_s \times \overline{X}_\pi] \in CH^s(\overline{X}_\pi \times \overline{X}_\pi)$ *is defined over* F.

This lemma implies the theorem. We have

$$i^*([\overline{X}_\pi \times c_s] + [c_s \times \overline{X}_\pi]) = [\overline{X}_\alpha \times pt] + [pt \times \overline{X}_\alpha]$$

and since i is defined over F this means that $[\overline{X}_\alpha \times pt] + [pt \times \overline{X}_\alpha]$ is also defined over F.

Proof of the lemma. X_π is a Pfister quadric, i.e. X_π is split over $F(X_\pi)$ implies then *thatc*$_s$ is defined over $F(X_\pi)$.

$$\begin{array}{ccc} CH^p(X_\pi \times X_\pi) & \longrightarrow & CH^s(X_\pi \times F(X_\pi)) \\ \downarrow & & \downarrow \\ CH^s(\overline{X}_\pi \times \overline{X}_\pi) & \longrightarrow & CH^s(\overline{X}_\pi \times \overline{F}(X_\pi)) \end{array}$$

We have a cycle β such that $\beta \times \overline{F}(X_\pi) = c_s$. Using the basis described above, we see that

$$CH^s(X_\pi \times X_\pi) \cong \oplus_{i=0}^s CH^i(X_\pi) \otimes CH^{s-i}(X_\pi)$$

and so

$$\beta = \underbrace{ac_s \otimes [\overline{X}_\pi]}_{\mapsto ac_s} + \underbrace{b[\overline{X}_\pi \otimes c_s]}_{\mapsto 0} + \underbrace{\sum_{i=0}^{s} c_i h^i \otimes h^{s-i}}_{\mapsto c_s h_{\overline{F}(X_\pi)}}$$

Where the map is j^* for $j : Spec\ \overline{F}(X_\pi) \to \overline{X_\pi}$.

$$CH^s(\overline{X}_\pi \times \overline{X}_\pi) \ni \underbrace{\beta}_{\text{defined over } F} = c_s \otimes [\overline{X}_\pi] + b[\overline{X}_\pi \otimes c_s] + \underbrace{\sum_{i=0}^{s} c_i h^i \otimes h^{s-i}}_{\text{defined over } F}$$

$2c_s$ is defined over F and so $2([\overline{X}_\pi] \times c_s)$ is also defined over F. If b is odd then $[\overline{X}_\pi \times c_s] + [c_s \times \overline{X}_\pi]$ is defined over F by addition or subtraction of copies of $2([\overline{X}_\pi] \times c_s)$. If b is even then $c_s \times [\overline{X}_\pi]$ is defined over F and so $(\ ,\ )^\tau \leftarrow [\overline{X}_\pi] \times c_s$ is defined over F. So $c_s \times [\overline{X}_\pi] + [\overline{X}_\pi] \times c_s$ is defined over F. □

□

Corollary 5. *The Rost motive M_α is also a direct summand of X_π in $Chow(F)$ (More generally $X_\pi \cong \oplus_{i=0}^{2^{n-1}-1} M_\alpha(i)$.*

$$\overline{M}_\alpha \cong \mathbb{1} \oplus \mathbb{L}^{\overbrace{2^{n-1}-1}_{\dim X_\alpha}}$$

This also appears in the motive of $\overline{X}_\pi$ correpsonding to $[X_\pi \times p] + [c_s \times h^s]$. The projector $[X_\pi \times p]$ is defined over the base and it is equal to $(i \times i)_*([\overline{X}_\alpha \times pt] + [pt \times \overline{X}_\alpha])([c_s \times \overline{X}_\pi] + [\overline{X}_\pi \times c_s])$.

We always have a map $\mathbb{L}^{2^{n-1}-1} \overset{\Psi^*}{\to} M_\alpha \overset{\Psi_*}{\to} \mathbb{Z}$ and this is a split exact sequence over $\overline{F}$ in particuler $\Psi^* \neq 0 \neq \Psi_*$.

$$X_\alpha(2^{n-1}-1) \overset{id_{X_\alpha} \otimes \Psi^*}{\to} X_\alpha \otimes M_\alpha \overset{id_{X_\alpha} \otimes \Psi_*}{\to} X_\alpha$$

becomes split exact over any point of X_α since X_α is a Pfister quadric. In particuler over the generic point.

$$X_\alpha(2^{n-1}-1) \overset{id_{X_\alpha} \otimes \Psi^*}{\to} X_\alpha \otimes M_\alpha \overset{id_{X_\alpha} \otimes \Psi_*}{\to} X_\alpha$$

is an exact triangle by a result of Voevodsky.

Let X be a smooth scheme. We can consider the simplicial complex

$$\ldots \overrightarrow{\overrightarrow{\overrightarrow{\rightarrow}}} X \times X \times X \overrightarrow{\overrightarrow{\rightarrow}} X \times X \overrightarrow{\rightarrow} X$$

Via the Dold-Kan construction this gives a motive $\check{C}(X) \in DM_-^{eff}(F)$ which we call the simplicial motive of X. There is a canonical map $\check{C}(X) \to \mathbb{Z}$ and this induces

$$\hom(M, \check{C}(X)) \to \hom(M, \mathbb{Z}) \text{ and } M \otimes \check{C}(X) \to M$$

for all $M \in DM_-^{eff}(F)$. If M is in the localising subcategory generated by the single object X, then both of these maps are isomorphisms.

Theorem 6. *There is an exact triangle in* $DM_-^{eff}(F)$

$$\begin{array}{ccccc}
M(\check{C}(X_\alpha))\{2^{n-1}-1\} & \xrightarrow{\gamma} & M_\alpha & \xrightarrow{\exists\epsilon} & M(\check{C}(X_\alpha)) \\
\downarrow & \nearrow & & \searrow & \downarrow \\
\mathbb{Z}\{2^{n-1}-1\} & & & & \mathbb{Z}
\end{array}$$

$$\begin{array}{ccccc}
M(\check{C}(X_\alpha))\{2^{n-1}-1\} & \xrightarrow{\gamma} & M_\alpha & \xrightarrow{\exists\epsilon} & Cone(\gamma) \\
 & & & \searrow^{\epsilon} & \downarrow{\scriptstyle \exists\phi} \\
 & & & & M(\check{C}(X_\alpha))
\end{array}$$

We have to show that ϕ is an isomorphism but since $Cone(\gamma)$ and $M(\check{C}(X_\alpha))$ are in the localising subcategory generated by X_α we have $Cone(\gamma)\otimes M(\check{C}(X_\alpha)) \cong Cone(\gamma)$ and $M(\check{C}(X_\alpha)) \otimes M(\check{C}(X_\alpha)) \cong M(\check{C}(X_\alpha))$. Therefore it is enough to show that $id_{M(\check{C}(X_\alpha))} \otimes \phi : M(\check{C}(X_\alpha)) \otimes Cone(\gamma) \to M(\check{C}(X_\alpha)) \otimes M(\check{C}(X_\alpha))$ has trivial cone. For this it is enough to show that $M(\check{C}(X_\alpha))$ is in the localising subcategory generated by X_α that $id_{X_\alpha}\otimes\phi$ has trivial cone in $DM_-^{eff}(F)$, which is implied by the exact triangle $X_\alpha(2^{n-1}-1) \to M_\alpha \to X_\alpha$.

2 A. Asok/ A. S. Merkurjev - Proof of the Milnor conjecture

Recall the motivic Hilbert 90 conjecture.

Motivic Hilbert 90 Conjecture.

$MH90(w,\ell)$ $H_{et}^{w+1,w}(E,\mathbb{Z}_{(\ell)}) = 0.$

If we have $MH90(w,\ell)$ for all finitely generated field extensions E of F then it follows that we have $BK(w,\ell$ for all $i \leq w$. We showed this by showing that motivic Hilbert 90 implies Beilinson-Lichtenbaum.

We want to show when $\ell = 2$ that $H_{et}^{w+1,w}(E,\mathbb{Z}_{(\ell)}) \to H_{et}^{w+1,w}(E(X_\alpha),\mathbb{Z}_{(\ell)})$ is injective when X_α is a small Pfister quadric. The goal of this lecture is to construct a group which will control the kernel of this map and the eventual goal is to show that this group is zero.

Recall that we have the motivic complexes $\mathbb{Z}(n)$ and the definitions

$$\begin{aligned}
B_\ell(w) &= \tau^{\leq w} Ra_* a^* \mathbb{Z}/\ell(q) \\
L(w) &= \tau^{\leq w} Ra_* a^* \mathbb{Z}(q) \\
K(w) &= Cone(\mathbb{Z}(w) \to L(w))
\end{aligned}$$

We will also write $K(w)_{(\ell)} = K(w) \otimes \mathbb{Z}_{(\ell)}$.

Proposition 7. $MH90(w,\ell)$ *implies that* $K(w)_{(\ell)}$ *is quasi-isomorphic to zero.*

Proof.

1. $H^{w+1}(K(w)_{(\ell)}(E) = 0$ by the assumption $MH90(w, \ell)$.
2. To show $H^{w+1}(K(w)_{(\ell)} = 0$ for all $i \leq w$ it suffices to show that $H^{i,w}(X, \mathbb{Z}_{(\ell)}) \to H^{i,w}_{et}(X, \mathbb{Z}_{(\ell)})$ is an isomorphism.

We will use the coefficients sequence

$$0 \to \mathbb{Z}_{(\ell)} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}_{(\ell)} \to 0$$

to reduce the statement for $\mathbb{Z}_{(\ell)}$ to the statements with $\mathbb{Q}$ coefficients and with $\mathbb{Z}/\ell^n$ coefficients. For $\mathbb{Q}$ coefficients the statement follows from what we saw last time. with $\mathbb{Z}/\ell^n$ coefficients it comes from $MH90(w, \ell) \implies BL(w, \ell)$. □

There are some extensions of this that we will have to use. Suppose $\mathcal{X}$ is a smooth simplicial scheme. We can define motivic cohomology of $\mathcal{X}$ in either the Nisnevich or étale topology. We will need the following corollary.

Corollary 8. *If $MH90(w, \ell)$ holds for all finitely generated E/F and if $\mathcal{X}$ is a smooth simplicial scheme then*

1. *$H^{p,q}(X, \mathbb{Z}_{(\ell)}) \to H^{p,q}_{et}(X, \mathbb{Z}_{(\ell)})$ is a monomorphism for $p - 2 \leq q \leq w$ and an isomorphism for $p - 1 \leq q \leq w$,*
2. *$H^{p,q}(X, \mathbb{Z}/\ell^n) \to H^{p,q}_{et}(X, \mathbb{Z}/\ell^n)$ is a monomorphism for $p - 1 \leq q \leq w$ and an isomorphism for $p \leq q \leq w$.*

We wil use the following spectral sequence which doesn't always converge, but there are arguments to get around this, and we won't discuss them.

$$E_1^{p,q} = \mathbb{H}^q(\mathcal{X}_p, \mathcal{F}) \implies \mathbb{H}^{p,q}(\mathcal{X}, \mathcal{F})$$

Definition. A presheaf F on Sm_F is birational if for any open dense immersion $U \to X$ the map $F(X) \to F(U)$ is a bijection.

Lemma 9. *If F is a birational and $\mathbb{A}^1$ invariant presheaf and $f : X \to Y$ is dominant and $X_{k(Y)}$ is $k(Y)$-rational, then $F(Y) \to F(X)$ is an isomorphism.*

Proof. Consider the sequare

$$\begin{array}{ccc} F(Y) & \xrightarrow{(1)} & F(k(Y)) \\ \downarrow & & \downarrow {\scriptstyle (3)} \\ F(X) & \xrightarrow[(2)]{} & F(k(X)) \end{array}$$

(1) and (2) are isomorphisms by birationality and (3) is an isomorphism by birationality and $\mathbb{A}^1$ homotopy invariance. □

Proposition 10. *Assume $MH90(w-1,\ell)$ holds for all E/F finitely generated. Then the presheaves $U \mapsto \mathbb{H}^*_{Nis}(U, K(w)_{(\ell)})$ are birational and $\mathbb{A}^1$ invariant.*

Proof. We will proceed by a series of reductions. We want to show $\mathbb{H}^*_{Nis}(X, K(w)_{(\ell)}) \to \mathbb{H}^*_{Nis}(U, K(w)_{(\ell)})$ is an isomorphism for $U \to X$. We asume that F is perfect. We also assume that $U = X - Z$ for Z a smooth closed subscheme with trivial normal bundle.

In this case we have a homotopy cofiber sequence $X - Z \to X \to X/X - Z \cong Th(\nu_{Z/X}) \cong Th(\mathcal{O}^c Z) \cong Z_+ \wedge T^{\wedge c}$ which gives rise to a long exact sequence

$$\mathbb{H}^*(Z_+ \wedge T^c, K(i)_{(\ell)}) \to \mathbb{H}^*(X, K(i)_{(\ell)}) \to \mathbb{H}^*(X - Z, K(i)_{(\ell)}) \to \mathbb{H}^*(\Sigma^1_s Z_+ \wedge T^c, K(i)_{(\ell)})$$

and we make the association $\mathbb{H}^*(Z_+ \wedge T^c, K(i)_{(\ell)}) \cong \mathbb{H}^{*-c}(Z_+ \wedge \mathbb{G}_m^{\wedge c}, K(i)_{(\ell)}) \cong \mathbb{H}^{*-c}(Z, K(i)_{(\ell)})_{-c}$. We want to show for any smooth scheme X and $c > 0$ that $\mathbb{H}^*(X, K(i)_{(\ell)})_{-c} = 0$. □

2.1 Interlude on contractions

Remember that

$$\begin{aligned} \hom(F \otimes \mathbb{G}_m, F \otimes \mathbb{G}_m) &\cong \hom(F, \underline{Rhom}(\mathbb{G}_m, F \otimes \mathbb{G}_m)) \\ &\cong \hom(F, F(1)[1]_{-1}) \end{aligned}$$

and the identity corresponds to some morphism $B : F[-1] \to F(1)_{-1}$. We also know that $H^i(F)_{-1} \cong H^i(F_{-1})$.

Example 11.

1. $(\mathbb{Z})_{-1} = 0$
2. $(\mathbb{Z}(1))_{-1} = \mathbb{Z}(i-1)[-1]$
3. $(\mathbb{Z}(n))_{-c} = \begin{cases} 0 & c > n \\ \mathbb{Z}(n-c)[-c] & c \leq n \end{cases}$
4. $(\mathbb{Z}/\ell(n))_{-c} = \begin{cases} 0 & c > n \\ \mathbb{Z}/\ell(n-c)[-c] & c \leq n \end{cases}$
5. $(R\alpha_*\alpha^*\mu_\ell^{\otimes n})_{-1} \cong (R\alpha_*\alpha^*\mu_\ell^{\otimes n-1})[-1]$.

From this we can see that $BL(w,\ell) \implies BL(w-1,\ell)$. Back to our problem,

$$\mathbb{H}^i(X, K(w)_{(\ell)})_{-1}$$

it suffices to show that

$$\mathbb{H}^i(X, \mathbb{Z}_{(\ell)}(w))_{-1} \to \mathbb{H}^i_{et}(X, \mathbb{Z}_{(\ell)}(w))_{-1}$$

is an isomorphism. We just jave to check this for $i \leq w+1$. Using $\mathbb{Z}_{(\ell)} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}_{(\ell)}$ we reduce to the statement for $\mathbb{Q}$-coefficients and $\mathbb{Q}/\mathbb{Z}_{(\ell)}$ coefficients.

$$\begin{array}{ccc}\mathbb{H}^{i,w}(X,\mathbb{Z}_{(\ell)})_{-1} & \longrightarrow & \mathbb{H}^{i,w}_{et}(X,\mathbb{Z}_{(\ell)})_{-1} \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{H}^{i-1,w-1}(X,\mathbb{Z}_{(\ell)})_{-1} & \longrightarrow & \mathbb{H}^{i-1,w-1}(X,\mathbb{Z}_{(\ell)})_{-1}\end{array}$$

The vertical isomorphisms are suspension isomorphisms in motivic cohomology, and the lower map is an isomorphism by $MH90(w-1,\ell) \implies BL$.

In the last talk we saw $\check{C}(X)$ introduced where X is a smooth F-scheme and $\check{C}(X)_q = \underbrace{X \times \cdots \times X}_{q+1 \text{ times}}$. If X has an F point, then the structure map $\check{C}(X) \to$ *Spec* F is a weak equivalence (this is checked on stalks). More generally, if $\hom(Y, X)$ is non-empty, then the product map $\check{C}(X) \times Y \to Y$ is a weak equivalence. There is an étale version of the motivic homotopy category and if X is non-empty then $\check{C}(X) \to$ *Spec* F is still a weak equivalence in this étale version.

Corollary 12. *Suppose X is a generically rational smooth F-scheme. If $MH90(w-1,\ell)$ holds then the maps*

$$\mathbb{H}^*(\check{C}(X), K(w)_{(\ell)}) \overset{(1)}{\to} \mathbb{H}^*(X, K(w)_{(\ell)}) \overset{(2)}{\to} \mathbb{H}^*(F(X), K(w)_{(\ell)})$$

are isomorphisms.

Sketch of proof. The map (2) is an isomorphism by birationality. For (1) consider the simplicial spectral sequence

$$E_1^{p,q} = \mathbb{H}^q(\check{C}(X)_p, K(w)_{(\ell)}) \implies \mathbb{H}^{p,q}(\check{C}(X), K(w)_{(\ell)})$$

The E_1 differentials d_1 are induced by projections from $X^{\times p+1} \to X^{\times p}$ but these maps are weak equivalences and so all the nontrivial differentials are isomorphisms and the spectral sequence degenerates. □

Corollary 13. *Suppose that X is a generically rational smooth simplicial F-scheme and $MH90(w-1,\ell)$ holds. Then there is an exact sequence*

$$\mathbb{H}^{w+1,w}(\check{C}(X), \mathbb{Z}_{(\ell)}) \to \mathbb{H}^{w+1,w}_{et}(F, \mathbb{Z}_{(\ell)}) \to H^{w+1,w}_{et}(F(X), \mathbb{Z}_{(\ell)})$$

Proof. We have the triangle

$$\mathbb{Z}_{(\ell)}(w) \to L_{(\ell)}(w) \to K(w)_{(\ell)}$$

which induces the diagram

$$\begin{array}{ccccc}\mathbb{H}^{w+1,w}((X),\mathbb{Z}_{(\ell)}) & \longrightarrow & \mathbb{H}^{w+1,w}_{et}(\check{C}(X),\mathbb{Z}_{(\ell)}) & \longrightarrow & \mathbb{H}^{w+1}(\check{C}(X),K(w)\mathbb{Z}_{(\ell)}) \\ \downarrow & & \downarrow & & \downarrow * \\ \mathbb{H}^{w+1,w}(F(X),\mathbb{Z}_{(\ell)}) & \longrightarrow & \mathbb{H}^{w+1,w}_{et}(F(X),\mathbb{Z}_{(\ell)}) & \longrightarrow & \mathbb{H}^{w+1,w}(F(X),K(w)\mathbb{Z}_{(\ell)})\end{array}$$

The map $*$ is an isomorphism by the previous corollary and the group in the bottom left corner is zero. We also have an isomorphism $\mathbb{H}_{et}(\check{C}(X), \mathbb{Z}_{(\ell)}) \cong \mathbb{H}_{et}^{w+1,w}(F, \mathbb{Z}_{(\ell)})$ so the result follows. □

References

[Ros98] M. Rost. The motive of a pfister form. http://www.math.uni-bielefeld.de/~rost/data/motive.pdf, 1998.