Steenrod operations

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Goals:

- Construct stable operations

$$P^i \colon H^{p,q}(\mathcal{X}) \to H^{p+2i,q+i}(\mathcal{X})$$

  for all spaces $\mathcal{X} \in \mathcal{H}(k)$ where $H^{p,q}$ denotes motivic cohomology with $\mathbf{Z}/2\mathbf{Z}$ coefficients.
- Study the motivic Steenrod algebra (generated over $H^{\star,\star}(k)$ by these P^i and the Bockstein) and its dual.
- Construct operations $Q_i \colon H^{p,q} \to H^{p+2^{i+1}-1,q+2^i-1}$ such that $Q_i \circ Q_i = 0$ ($\Rightarrow$ definition of Margolis homology).
- Understand the action of the Steenrod algebra on Thom classes.

We fix a (perfect) base field k. We assume its characteristic is not two.

Definition

For $p \geq q \geq 0$, the motivic sphere $S^{p,q}$ is $S^{p-q} \wedge \mathbf{G}_{\mathrm{m}}^{\wedge q} \in \mathcal{H}_{\bullet}(k)$.

We have a tautological class in $\widetilde{H}^{p,q}(S^{p,q})$ that induces isomorphisms:

$$\widetilde{H}^{a,b}(\mathcal{X}) \xrightarrow{\sim} \widetilde{H}^{a+p,b+q}(S^{p,q} \wedge \mathcal{X})$$

Definition

A stable cohomological operation of bidegree (a,b) is family of natural transformations $\widetilde{H}^{i,j}(\mathcal{X}) \to \widetilde{H}^{i+a,j+b}(\mathcal{X})$ for $X \in \mathcal{H}_{\bullet}(k)$ such that the action on $\widetilde{H}^{i-p,j-q}$ is determined by the action on $\widetilde{H}^{i,j}$ through the identifications

$$\widetilde{H}^{i-p,j-q}(\mathcal{X}) = \widetilde{H}^{i,j}(S^{p,q} \wedge \mathcal{X})$$

Lemma

One can (re)construct a unique stable operation for the datum of the action on $\widetilde{H}^{2n,n}$ for $n \geq 0$ provided they are compatible with the identification $\widetilde{H}^{2n,n}(\mathcal{X}) \xrightarrow{\sim} \widetilde{H}^{2(n+1),n+1}(S^{2,1} \wedge \mathcal{X})$. (Note that $S^{2,1} \simeq \mathbf{P}^1$.)

(Let Λ be $\mathbf{Z}/2\mathbf{Z}$.)
For all $(p,q) \in \mathbf{Z}^2$, we have motivic Eilenberg-Mac Lane spaces $K(\Lambda(q),p) \in \mathcal{H}_\bullet(k)$, i.e.,

$$\widetilde{H}^p(\mathcal{X},\Lambda(q)) = \widetilde{H}^{p,q}(\mathcal{X}) \simeq \mathrm{Hom}_{\mathcal{H}_\bullet(k)}(\mathcal{X}, K(\Lambda(q),p))$$

Yoneda's lemma $\Rightarrow$ a natural transformation $\widetilde{H}^{i,j}(\mathcal{X}) \to \widetilde{H}^{i+a,j+b}(\mathcal{X})$ for $X \in \mathcal{H}_\bullet(k)$ is the same as a morphism $K(\Lambda(j),i) \to K(\Lambda(j+b), i+a)$ in $\mathcal{H}_\bullet(k)$.

Then, a stable cohomology operation is the same a family of maps $f_n\colon K(\Lambda(n),2n) \to K(\Lambda(n+b), 2n+a)$ in $\mathcal{H}_\bullet(k)$ such that the following diagram commute:

$$\begin{array}{ccc}
K(\Lambda(n),2n) & \xrightarrow{f_n} & K(\Lambda(n+j),2n+i) \\
\downarrow \sim & & \downarrow \sim \\
\Omega_{\mathbf{P}^1} K(\Lambda(n+1),2n+2) & \xrightarrow{\Omega_{\mathbf{P}^1}(f_{n+1})} & \Omega_{\mathbf{P}^1} K(\Lambda(n+j+1),2n+2+i)
\end{array}$$

This is essentially the way we shall define the operations P^i.

Main source:

Vladimir Voevodsky. Reduced power operations in motivic cohomology. *Publications Mathématiques de l'IHÉS* 98 (2003), pages 1–57.

1 Construction of Steenrod operations

2 Properties of the Steenrod operations

3 The Steenrod algebra and its dual

4 Applications

Definition

Let $X \to S$ be a smooth morphism in Sm/k. $c_{\mathrm{equi}}(X/S, 0)$ is the free Λ-module generated by integral closed subschemes Z in X such that $Z \to S$ is a finite morphism and a surjection over a connected component of S. (There is a fonctoriality associated to a base change $S' \to S$.)

Definition

Let $X \in Sm/k$. $\Lambda_{\mathrm{tr}}(X)$ is the sheaf of groups over Sm/k (for the Nisnevich topology) defined by $\Lambda_{\mathrm{tr}}(X)(U) = c_{\mathrm{equi}}(U \times_k X/U, 0)$.
For any $i \geq 0$, K_i is the underlying sheaf of sets of the sheaf of abelian groups $\Lambda_{\mathrm{tr}}(\mathbf{A}^i)/\Lambda_{\mathrm{tr}}(\mathbf{A}^i - \{0\})$. This is the Eilenberg-Mac Lane space $K(\Lambda(i), 2i) \in \mathcal{H}_\bullet(k)$.

Definition

Let E be a vector bundle of rank r on $X \in Sm/k$. We denote $\mathrm{Th}_X\, E = E/E - \{0\} \simeq \mathbf{P}(E \oplus \mathscr{O}_X)/\mathbf{P}(E)$ the Thom space of X.

Proposition

$\widetilde{H}^{\star,\star}(\mathrm{Th}_X\, E)$ is a free $H^{\star,\star}(X)$-module of rank 1 generated by the Thom class $t_E = \xi^r + c_1(E)\xi^{r-1} + \cdots + c_r(E) \in \ker(H^{\star,\star}(\mathbf{P}(E \oplus \mathscr{O}_X)) \to H^{\star,\star}(\mathbf{P}(E))) \simeq \widetilde{H}^{\star,\star}(\mathrm{Th}_X\, E)$ where $\xi = c_1(\mathscr{O}(1)) \in H^{2,1}(\mathbf{P}(E \oplus \mathscr{O}_X))$.

Definition

The Euler class of E in $H^{2r,r}(X)$ is the image of t_E by the restriction map $\widetilde{H}^{\star,\star}(\mathrm{Th}_X\, E) \to H^{\star,\star}(X)$ induced by the zero section $X \to \mathrm{Th}_X\, E$. This class is the highest Chern class $c_r(E)$.

Lemma

If $E \to F$ is an admissible monomorphism of vector bundles on X, the image of t_F by the restriction map $\widetilde{H}^{\star,\star}(\mathrm{Th}_X\, F) \to \widetilde{H}^{\star,\star}(\mathrm{Th}_X\, E)$ induced by the obvious morphism $\mathrm{Th}_X\, E \to \mathrm{Th}_X\, F$ is $t_E \cdot c_r(F/E)$ where r is the rank of F/E.

Lemma

If $E \to F$ is an admissible monomorphism of vector bundles on X, the image of t_F by the restriction map $\widetilde{H}^{\star,\star}(\mathrm{Th}_X\, F) \to \widetilde{H}^{\star,\star}(\mathrm{Th}_X\, E)$ induced by the obvious morphism $\mathrm{Th}_X\, E \to \mathrm{Th}_X\, F$ is $t_E \cdot c_r(F/E)$ where r is the rank of F/E.

Proof.

Let e be the rank of E. We denote $\xi = c_1(\mathscr{O}(1))$ on various projective bundles. Because of the relations $c_i(E \oplus \mathscr{O}_X) = c_i(E)$, we have the following identity in $H^{\star,\star}(\mathbf{P}(E \oplus \mathscr{O}_X))$:

$$\xi^{e+1} + c_1(E)\xi^e + \cdots + c_e(E)\xi = 0 \qquad i.e., \quad t_E\xi = 0 \,.$$

Multiplicativity of the Chern polynomial for E and F/E gives:

$$t_F = (\xi^e + c_1(E)\xi^{e-1} + \cdots + c_e(E)) \cdot (\xi^r + c_1(F/E)\xi + \cdots + c_r(F/E))$$

This is in $H^{\star,\star}(\mathbf{P}(F \oplus \mathscr{O}_X))$. Restricted to $\mathbf{P}(E \oplus \mathscr{O}_X)$, we obtain :

$$t_E \cdot ((\ldots) \cdot \xi + c_r(F/E)) = t_E \cdot c_r(F/E)$$

□

The last proposition says that $\mathrm{Th}_X E$ and $S^{2r,r} \wedge X_+$ have the same cohomology. More precisely, they have the same *motive*. The following corollary is even more precise as it states something relative to X:

Corollary

Let $X \in Sm/k$. (We denote $a \colon X \to \operatorname{Spec} k$ the projection.) Let E be a vector bundle over X of rank r. We define the sheaf of sets $KM(\mathrm{Th}_X E)$ induced by the sheaf of abelian groups over Sm/X associated to the presheaf

$$U \longmapsto c_{equi}(U \times_X E/U, 0)/c_{equi}(U \times_X (E - \{0\}), 0))$$

Then, the Thom class t_E induces an isomorphism in $\mathcal{H}_\bullet(X)$:

$$KM(\mathrm{Th}_X E) \xrightarrow{\sim} KM(\mathrm{Th}_X \mathbf{A}^r) = a^\star K_r \ .$$

("KM" should be thought as a composition of two adjoint functors. M is the "motive" functor from spaces to motives, and K is its right adjoint, that forgets transfers and abelian groups structures on sheaves.) Roughly, the only difficulty here is how t_E induces a map. Then, it is quite obvious that it is an isomorphism.

Data:

- G is a finite group;
- $r\colon G \to \mathfrak{S}_n$ is a morphism, i.e., essentially a (left-)action of G on a finite set X with n elements ;
- $U \in Sm/k$ is equipped with a free (left-)action of G.

To this, we shall attach a cohomological operation for all $i \geq 0$:

$$P\colon \widetilde{H}^{2i,i}(\mathcal{X}) \to \widetilde{H}^{2in,in}(\mathcal{X} \wedge (G\backslash U)_+) .$$

Then, we will apply it to the case U is the open subset of a big enough (faithful) linear representation $G \to \mathrm{GL}(V)$ on which G acts freely, so that $G\backslash U$ is an approximation of the geometric classifying space $\mathbf{B}_{\mathrm{gm}}G$. When we understand the motive of $\mathbf{B}_{\mathrm{gm}}G$, we will be able to define the expected Steenrod operations.

We linearise the action of G on $X = \{1, \dots, n\}$ as a k-linear action of G on $V = k^n \simeq \oplus_{x \in X} k \cdot e_x$ with $g.e_x = e_{g.x}$. This defines an action of G on the affine space $\mathbf{A}^n$.

Proposition

The quotient scheme $G \backslash (U \times \mathbf{A}^n)$ of $U \times \mathbf{A}^n$ by the product action of G is a vector bundle ξ of rank n over $G \backslash U$.

Assume for simplicity that $U = \operatorname{Spec} A$ is affine. We have a right-action of G on A (denoted $g^\star f$ for $f \in A$). We equip $M = A \otimes_k V$ with a *semilinear* left-action $g.(a \otimes v) = (g^{-1\star} a) \otimes (g.v)$.
The subgroup $M_0 = M^G$ of elements fixed by G is a module over the algebra A^G of functions over U fixed under the action of G. By definition, $G \backslash U = \operatorname{Spec} A^G$. The theory of faithfully flat descent implies that the canonical map of A-modules

$$M_0 \otimes_{A^G} A \to M$$

is an isomorphism. As the A^G-algebra A is faithfully flat, properties of M_0 over A^G reflects those on M over A. This implies that M_0 is a projective module of rank n over A^G. Then, $G \backslash (U \times \mathbf{A}^n) = \operatorname{Spec} \mathbf{S}^\star_{A^G} M_0^\vee$, so that ξ is a vector bundle (which is is self-dual).

Proposition

For all $i, j \geq 0$, we have a canonical pairing in the category of pointed sheaves over Sm/k:

$$K_i \wedge K_j \to K_{i+j}$$

We know that $K_n(Y) = c_{\text{equi}}(Y \times \mathbf{A}^n/Y, 0)/c_{\text{equi}}(Y \times (\mathbf{A}^n - \{0\})/Y, 0)$.
The pairing is induced by the obvious product map:

$$c_{\text{equi}}(Y \times \mathbf{A}^i/Y, 0) \times c_{\text{equi}}(Y \times \mathbf{A}^j/Y, 0) \to c_{\text{equi}}(Y \times \mathbf{A}^{i+j}/Y, 0)$$

given by the external product of cycles followed by the base change by the diagonal $Y \to Y \times Y$.

Corollary

For any $i \geq 0$, we have a "raising to the power n" map:

$$K_i \to K_{in}$$

that is $\mathfrak{S}_n$-equivariant for the trivial action on K_i and the action on $K_{in} \simeq KM(\mathrm{Th}_k\, V^{\oplus i})$ where $V = k^n$ is the permutation representation as before.

Composing this morphism $K_i \to K_{in}$ with the "constant function morphism" $K_{in} \to \mathbf{Hom}(U, K_{in})$, we get a morphism:

$$K_i \to \mathbf{Hom}(U, K_{in})$$

The $\mathfrak{S}_n$-equivariance property stated before implies that this factors through the subsheaf of $\mathbf{Hom}_G(U, K_{in})$ of G-equivariant morphisms. More precisely, the image of an element on $K_i(Y)$ induced by an element of $c_{\text{equi}}(Y \times \mathbf{A}^i/Y, 0)$ shall be an element in the group on the right:

$$c_{\text{equi}}(Y \times G\backslash(U \times \mathbf{A}^{in})/Y \times G\backslash U, 0) \xrightarrow{\sim} c_{\text{equi}}(Y \times U \times \mathbf{A}^{in}/Y \times U, 0)^G$$

This isomorphism comes from the étale descent of cycles. Then on the left, we recognise $c_{\text{equi}}(Y \times \xi^{\oplus i}/Y \times G\backslash U, 0)$. If $a\colon G\backslash U \to \operatorname{Spec} k$ is the projection, we have defined the first morphism in the following composition in $\mathcal{H}_\bullet(k)$:

$$K_i \to a_\star KM(\mathrm{Th}_{G\backslash U}\, \xi^{\oplus i}) \to \mathbf{R}a_\star KM(\mathrm{Th}_{G\backslash U}\, \xi^{\oplus i}) \simeq \mathbf{R}a_\star a^\star K_{in} \simeq \mathbf{R}\,\mathbf{Hom}(G\backslash U, K_{in})$$

We have defined the total operation:

$$K_i \to \mathbf{R}\,\mathbf{Hom}(G\backslash U, K_{in}) \quad \overset{\text{id est}}{\longleftrightarrow} \quad P\colon K_i \wedge (G\backslash U)_+ \to K_{in}$$

This morphism $P\colon K_i \wedge (G\backslash U)_+ \to K_{in}$ in $\mathcal{H}_\bullet(k)$ induces a cohomology operation:

$$P\colon \widetilde{H}^{2i,i}(\mathcal{X}) \to \widetilde{H}^{2in,in}(\mathcal{X} \wedge (G\backslash U)_+)$$

for all $\mathcal{X} \in \mathcal{H}_\bullet(k)$.

Lemma

The composition

$$K_i \to K_i \wedge (G\backslash U)_+ \xrightarrow{P} K_{in}$$

where the first map is induced by a rational point of U is the "raising to the power n" morphism.

(To prove this lemma, one may for instance replace U by the orbit of the given rational point, in which case it is obvious.)
It means that if $x \in \widetilde{H}^{2i,i}(\mathcal{X})$, then $u^\star P(x) = x^n \in \widetilde{H}^{2in,in}(\mathcal{X})$ where u is the map $\mathcal{X} \to \mathcal{X} \wedge (G\backslash U)_+$ induced by a rational point of U.

Proposition

Let $\mathscr{X}$ and $\mathscr{Y}$ be two objects of $\mathcal{H}_\bullet(k)$, $x \in \widetilde{H}^{2i,i}(\mathscr{X})$ and $y \in \widetilde{H}^{2j,j}(\mathscr{Y})$. Then,

$$P(x \cup y) = \Delta^\star(P(x) \cup P(y))$$

in $\widetilde{H}^{2(i+j)n,(i+j)n}(\mathscr{X} \wedge \mathscr{Y} \wedge (G\backslash U)_+)$ where

$$\Delta\colon \mathscr{X} \wedge \mathscr{Y} \wedge (G\backslash U)_+ \to \mathscr{X} \wedge \mathscr{Y} \wedge (G\backslash U)_+^2$$

is induced by the diagonal of $G\backslash U$.

It follows from a very direct computation.

The Bockstein β is the cohomology operation that naturally fits into the following long exact sequences coming from the short exact sequence $0 \to \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$:

$$\cdots \to \widetilde{H}^{\star,\star}(\mathcal{X}, \mathbf{Z}/2) \to \widetilde{H}^{\star,\star}(\mathcal{X}, \mathbf{Z}/4) \to \widetilde{H}^{\star,\star}(\mathcal{X}, \mathbf{Z}/2) \xrightarrow{\beta} \widetilde{H}^{\star+1,\star}(\mathcal{X}, \mathbf{Z}/2) \to \dots$$

In particular, $\beta x = 0$ if and only if x lifts as a cohomology class with coefficients $\mathbf{Z}/4\mathbf{Z}$. (Also, $\beta \circ \beta = 0$ and $\beta(xy) = x\beta(y) + (\beta x)y$.)

Theorem

If $G = \mathbf{Z}/2\mathbf{Z}$ and $n = 2$, for any cohomology class $x \in \widetilde{H}^{2i,i}(\mathcal{X})$, we have:

$$\beta(P(x)) = 0$$

A rough idea of the proof is that there is a way to lift P as:

$$\widetilde{P} \colon K_{i,\mathbf{Z}/2} \to \mathbf{R}\,\mathbf{Hom}(G \backslash U, K_{i,\mathbf{Z}/4}) \ ,$$

the main remark is that in some sense, somewhere, $(x + 2y)^2 \equiv x^2 + 2(xy + yx) \mod 4$ and $xy + yx$ can be interpreted as a transfer of a certain cycle xy for the an action of $\mathbf{Z}/2$ by transposition.

The geometric classifying space of a linear algebraic group G is the colimit $\mathbf{B}_{\mathrm{gm}}G = \operatorname{colim} G\backslash U_n$ where U_n is the open subset of $V^{\oplus n}$ on which G acts freely and V is some faithful linear representation of G.
For $G = \mu_\ell$, we take $V = \mathbf{A}^1$ on which $\mu_\ell \subset \mathbf{G}_{\mathrm{m}}$ acts by multiplication. Then, $U_n = \mathbf{A}^n - \{0\}$.

Proposition

$\mathbf{B}_{gm}\mu_\ell$ *is the complement of the zero section of the line bundle* $\mathscr{O}(-\ell)$ *on* $\mathbf{P}^\infty$.

We have a projection $\mu_\ell\backslash(\mathbf{A}^n - \{0\}) \to \mathbf{G}_{\mathrm{m}}\backslash(\mathbf{A}^n - \{0\}) = \mathbf{P}^{n-1}$. Because of the short exact sequence

$$0 \to \mu_\ell \to \mathbf{G}_{\mathrm{m}} \xrightarrow{x \mapsto x^\ell} \mathbf{G}_{\mathrm{m}} \to 0 \,,$$

we see that this projection is a $\mathbf{G}_{\mathrm{m}}/\mu_\ell \xrightarrow{\sim} \mathbf{G}_{\mathrm{m}}$-torsor, which is obtained from the tautological $\mathbf{G}_{\mathrm{m}}$-torsor $\mathbf{A}^n - \{0\} \to \mathbf{P}^{n-1}$ (punctured universal line $\mathscr{O}(-1)$) by covariant functoriality associated to the morphism $\mathbf{G}_{\mathrm{m}} \xrightarrow{x \mapsto x^\ell} \mathbf{G}_{\mathrm{m}}$. Then, we get the punctured $\mathscr{O}(-1)^{\otimes \ell} = \mathscr{O}(-\ell)$.

Proposition

Let $X \in Sm/k$. Let L be a line bundle on X. We let $L - \{0\}$ be the punctured bundle, i.e., the complement of the zero section $s\colon X \to L$. Then, we have a distinguished triangle in $DM_-^{eff}(k)$:

$$M(L - \{0\}) \to M(X) \to M(X)(1)[2] \xrightarrow{+}$$

where the map $M(X) \to M(X)(1)[2]$ is the multiplication by $c_1(L)$.

Proof.

It comes from the distinguished triangle $M(L - \{0\}) \to M(L) \to \widetilde{M}(\mathrm{Th}_X\, L) \xrightarrow{+}$ and the isomorphism $\widetilde{M}(\mathrm{Th}_X\, L) \simeq M(X)(1)[2]$ induced by the Thom class. Then, the composition $M(X) \xrightarrow{\sim} M(L) \to \widetilde{M}(\mathrm{Th}_X\, L)$ is identified with the multiplication with the Euler class of L, i.e., $c_1(L)$. □

Proposition

Assume now that the line bundle L on X is such that $c_1(L) = 0 \in H^{2,1}(X)$ (for a certain coefficient ring Λ), then there exists a class $u \in H^{1,1}(L - \{0\}, \Lambda)$ (well defined modulo the image of $H^{1,1}(X, \Lambda)$), such that the projection $L - \{0\} \to X$ and the classes 1 and u induce an isomorphism:

$$M(L - \{0\}) \xrightarrow{\sim} M(X) \oplus M(X)(1)[1]$$

The distinguished triangle reduces to a split short exact sequence in $DM_{-}^{\mathrm{eff}}(k)$:

$$0 \to M(X)(1)[1] \xrightarrow{\delta} M(L - \{0\}) \to M(X) \to 0$$

Then, applying the cohomological functor $H^{1,1}$, we obtain a class $u \in H^{1,1}(L - \{0\})$ (unique modulo $H^{1,1}(X)$) such that $\delta^\star(u) = 1 \in H^{0,0}(X)$. This u defines a map $M(L - \{0\}) \to M(X)(1)[1]$ which is a retraction of δ because δ is compatible with certain $M(X)$-comodule structures (this is related to saying that $\delta^\star$ is $H^{\star,\star}(X)$-linear, at least up to signs).

Corollary

For $\Lambda = \mathbf{Z}/\ell\mathbf{Z}$, we have a class $u \in H^{1,1}(\mu_\ell \backslash (\mathbf{A}^n - \{0\}))$ such that the projection to $\mathbf{P}^{n-1}$ and the classes 1 and u induce an isomorphism in $DM_-^{eff}(k; \mathbf{Z}/\ell\mathbf{Z})$:

$$M(\mu_\ell \backslash (\mathbf{A}^n - \{0\})) \xrightarrow{\sim} M(\mathbf{P}^{n-1}) \oplus M(\mathbf{P}^{n-1})(1)[1]$$

(Note that $c_1(\mathscr{O}(-\ell) = \ell c_1(\mathscr{O}(-1))$ which is zero modulo ℓ.) The class u from the previous proposition is made unique here by the condition that for one (or any) rational point x of $U_n = \mathbf{A}^n - \{0\}$, the restriction $x_{|[u]}$ is zero. This follows from the isomorphism $k^\times / k^{\times \ell} \simeq H^{1,1}(k) \xrightarrow{\sim} H^{1,1}(\mathbf{P}^{n-1}(k))$.

Proposition

For any $n \geq 0$, we have an isomorphism

$$M(\mathbf{P}^{n-1}) \xrightarrow{\sim} \bigoplus_{i=0}^{n-1} \Lambda(i)[2i]$$

that is induced by the classes $1, v, \ldots, v^{n-1}$ with $v = c_1(\mathscr{O}(1)) \in H^{2,1}(\mathbf{P}^{n-1})$.

Corollary

The obvious maps $M(\mathbf{P}^{n-1}) \to M(\mathbf{P}^n)$ *and* $M(\mu_\ell \backslash (\mathbf{A}^n - \{0\})) \to M(\mu_\ell \backslash (\mathbf{A}^{n+1} - \{0\}))$ *are split monomorphisms.*

This is so as to ensure there is no technical difficulties when taking colimits:

Corollary

The classes $1, v, v^2, \ldots$ *induce an isomorphism:*

$$M(\mathbf{P}^\infty) \xrightarrow{\sim} \oplus_{i \geq 0} \Lambda(i)[2i]$$

and the classes $1, u$ *and the projection* $\mathbf{B}_{gm}\mu_\ell \to \mathbf{P}^\infty = \mathbf{B}_{gm}\mathbf{G}_m$ *induce an isomorphism:*

$$M(\mathbf{B}_{gm}\mu_\ell) \xrightarrow{\sim} M(\mathbf{P}^\infty) \oplus M(\mathbf{P}^\infty)(1)[1]$$

It follows that if we want to understand the cohomology algebra of $\mathbf{B}_{\mathrm{gm}}\mu_\ell$, we have to compute $u^2 \in H^{2,2}(\mathbf{B}_{\mathrm{gm}}\mu_\ell)$.
Obviously, if $\ell \neq 2$, we have $u^2 = 0$. From now, we assume $\ell = 2$.

We define $\tau \in H^{0,1}(k) \simeq \mu_2(k)$ the element corresponding to $-1 \in k$ and $\rho \in H^{1,1}(k) \simeq k^\times/k^{\times 2}$ the class of -1. Note that $\beta(\tau) = \rho$.

Proposition

In $H^{2,2}(\mathbf{B}_{gm}\mathbf{Z}/2\mathbf{Z})$, we have $u^2 = \tau v + \rho u$.

Proof.

For degree reasons, it follows from the decomposition of the motive of $\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2\mathbf{Z}$, that u^2 writes uniquely as $u^2 = xv + yu + z$ with $x \in H^{0,1}(k)$, $y \in H^{1,1}(k)$ and $z \in H^{2,2}(k)$. The elements u, v and u^2 vanish when restricted to a suitable base-point of $\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2\mathbf{Z}$. This shows that $z = 0$.

The restriction to the cohomology of $\{\pm 1\}\backslash U_1 = \{\pm 1\}\backslash \mathbf{G}_{\mathrm{m}} \simeq \operatorname{Spec} k[t, t^{-1}]$ corresponds to removing the term xv. We use the fact that $H^{2,2}(\operatorname{Spec} k[t, t^{-1}]) \hookrightarrow H^{2,2}(\operatorname{Spec} k(t, t^{-1})) = K_2^M(k(t, t^{-1}))$. The image of u in $K_1^M(k(t, t^{-1})$ can be identified with $\{t\}$. Then, the result follows from $\{t, t\} = \{t, t\} - \{-t, t\} = \{-1, t\} = \{-1\} \cdot \{t\}$. Thus, $y = \rho$.
(If $k \subset \mathbf{C}$), the coefficient $x \in \mu_2(k)$ is either 0 or τ. One can see the difference by taking complex points and using the structure of the cohomology algebra modulo 2 of the group $\mathbf{Z}/2\mathbf{Z}$, in which $u^2 \neq 0$.

□

Proposition

In $H^{2,1}(\mathbf{B}_{gm}\mathbf{Z}/2\mathbf{Z})$, we have $\beta u = v$.

Proof.

For degree reasons, we have either $\beta u = 0$ or $\beta u = v$.

$$\begin{array}{ccccc} H^{1,1}(L-\{0\},\mathbf{Z}/4\mathbf{Z}) & \xrightarrow{\delta^\star} & H^{0,0}(X,\mathbf{Z}/4\mathbf{Z}) & \xrightarrow{\cdot c_1(L)} & H^{2,1}(X,\mathbf{Z}/4\mathbf{Z}) \\ \downarrow & & \downarrow & & \\ H^{1,1}(L-\{0\},\mathbf{Z}/2\mathbf{Z}) & \xrightarrow{\delta^\star} & H^{0,0}(X,\mathbf{Z}/2\mathbf{Z}) & \longrightarrow & 0 \end{array}$$

Assuming $\beta u = 0$, there is a lifting $\tilde{u}$ of u in $H^{1,1}(L-\{0\},\mathbf{Z}/4)$ (we take $X = \mathbf{P}^{n-1}$ for $n \geq 2$ and $L = \mathscr{O}(-2)$). Then $\delta^\star\tilde{u} = \pm 1$, then the image of $\tilde{u}$ in $H^{2,1}(\mathbf{P}^{n-1},\mathbf{Z}/4\mathbf{Z})$ is $\pm c_1(\mathscr{O}(-2)) = \pm 2c_1(\mathscr{O}(1)) \neq 0$ (modulo 4). We get a contradiction with the exactness of the first line. Then $\beta u = v$. □

Corollary

For any $X \in \mathcal{H}_\bullet(k)$, we have canonical isomorphisms of bigraded groups:

$$\begin{aligned}\widetilde{H}^{\star,\star}(\mathcal{X} \wedge (\mathbf{B}_{gm}\mathbf{Z}/2\mathbf{Z})_+) &\simeq \lim_n \widetilde{H}^{\star,\star}(\mathcal{X} \wedge (\{\pm 1\}\backslash(\mathbf{A}^n - \{0\}))_+) \\ &\simeq \widetilde{H}^{\star,\star}(\mathcal{X})[u,v]/(u^2 - \tau v - \rho u)\end{aligned}$$

Let $d \geq 0$. The construction P (for $i = d$ and $n = 2$) for the action of $\mathbf{Z}/2\mathbf{Z}$ on $\mathbf{A}^n - \{0\}$ for all $n \geq 1$ defines then a morphism for all $\mathcal{X} \in \mathcal{H}_\bullet(k)$:

$$P\colon \widetilde{H}^{2d,d}(\mathcal{X}) \to \widetilde{H}^{4d,2d}(\mathcal{X} \wedge (\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2\mathbf{Z})_+) \,.$$

Definition

We define cohomological operation $P^i\colon \widetilde{H}^{2d,d} \to \widetilde{H}^{2d+2i,d+i}$ (for $i \leq d$) and $B^i\colon \widetilde{H}^{2d,d} \to \widetilde{H}^{2d+2i+1,d+i}$ (for $i \leq d-1$) by the following relation for all $x \in \widetilde{H}^{2d,d}(\mathcal{X})$:

$$P(x) = \sum_{i \leq d} P^i(x) v^{d-i} + \sum_{i \leq d-1} B^i(x) u v^{d-1-i}$$

(We set $P^i = 0$ for $i > d$ and $B^i = 0$ for $i \geq d$.)

Proposition

- $B^i = \beta P^i$;
- $\beta B^i = 0$.

Proof.

Let $x \in \widetilde{H}^{2d,d}(\mathcal{X})$. We know that $\beta P(x) = 0$; $v = \beta(u)$, then $\beta(v^k) = 0$ and $\beta(uv^k) = v^{k+1}$:

$$\begin{aligned} \beta P(x) &= \beta\left(\sum_i P^i(x) v^{d-i} + \sum_i B^i(x) u v^{d-1-i}\right) \\ &= \sum_i (\beta P^i(x) + B^i(x)) v^{d-i} + \sum_i \beta B^i(x) u v^{d-1-i} \end{aligned}$$

□

We also define $\mathrm{Sq}^{2i} = P^i$ and $\mathrm{Sq}^{2i+1} = B^i$. The operation Sq^j shifts the first degree by j and the second degree by $\lfloor \frac{j}{2} \rfloor$.

Theorem

There is no nontrivial cohomology operation

$$\widetilde{H}^{2d,d} \to \widetilde{H}^{p,q}$$

for $q < d$ and for $q = d$, there are no nontrivial operation for $p < 2d$. The operations $\widetilde{H}^{2d,d} \to \widetilde{H}^{2d,d}$ are given by the multiplication by an element in $\mathbf{Z}/2\mathbf{Z}$.

Corollary

$\mathrm{Sq}^j = 0$ *for* $j < 0$.

Corollary

For $x \in \widetilde{H}^{2d,d}(\mathcal{X})$, $P(x) = \sum_{i=0}^{d} P^i(x)v^{d-i} + \sum_{i=0}^{d-1} B^i(x)uv^{d-1-i}$.

Proposition

We let $t \in \widetilde{H}^{2,1}(S^{2,1})$ $(S^{2,1} \simeq \mathbf{A}^1/(\mathbf{A}^1 - \{0\}))$ *be the tautological class. Then, for all* $i \geq 0$ *and* $x \in \widetilde{H}^{2d,d}(\mathcal{X})$, $P^i(x \cup t) = P^i(x) \cup t$ *and* $B^i(x \cup t) = B^i(x) \cup t$.

Lemma

In $\widetilde{H}^{4,2}(\mathbf{A}^1/(\mathbf{A}^1 - \{0\}) \wedge (\mathbf{B}_{gm}\mathbf{Z}/2\mathbf{Z})_+)$, *we have* $P(t) = t \cup v$.

This lemma implies the proposition using the formulas $P(x \cup t) = P(x) \cup P(t) = P(x) \cup t \cup v$ and identifying the different terms. To prove it, we shall use:

Lemma

We let $\delta\colon (\mathbf{B}_{gm}\mathbf{Z}/2\mathbf{Z})_+ \wedge (\mathbf{A}^1/\mathbf{A}^1 - \{0\}) \to \mathrm{Th}_{\mathbf{B}_{gm}\mathbf{Z}/2\mathbf{Z}}\,\xi$ *be the map on Thom spaces induces by the obvious inclusion* $\mathscr{O} \to \xi$ *of vector bundles on* $\mathbf{B}_{gm}\mathbf{Z}/2\mathbf{Z}$. *Then,* $P(t) = \delta^\star t_\xi$.

This is a very simple computation.

Lemma

In $\widetilde{H}^{4,2}(\mathbf{A}^1/(\mathbf{A}^1-\{0\})\wedge(\mathbf{B}_{gm}\mathbf{Z}/2\mathbf{Z})_+)$, we have $P(t)=t\cup v$.

We use:

Lemma

If $E\to F$ is an admissible monomorphism of vector bundles on X, the image of t_F by the restriction map $\widetilde{H}^{\star,\star}(\mathrm{Th}_X F)\to\widetilde{H}^{\star,\star}(\mathrm{Th}_X E)$ induced by the obvious morphism $\mathrm{Th}_X E\to\mathrm{Th}_X F$ is $t_E\cdot c_r(F/E)$ where r is the rank of F/E.

When we apply it to $\delta\colon(\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2\mathbf{Z})_+\wedge(\mathbf{A}^1/\mathbf{A}^1-\{0\})\to\mathrm{Th}_{\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2\mathbf{Z}}\,\xi$, we get:

$$P(t)=\delta^\star t_\xi=t\cup c_1(\xi/\mathscr{O})$$

Lemma

The bundle $\xi/\mathscr{O}$ identifies to the inverse image of $\mathscr{O}(\pm1)$ by the projection $\mathbf{B}_{gm}\mathbf{Z}/2\mathbf{Z}\to\mathbf{P}^\infty$.

It follows that $c_1(\xi/\mathscr{O})=v$.

Lemma

The bundle $\xi/\mathcal{O}$ identifies to the inverse image of $\mathcal{O}(\pm 1)$ by the projection $\mathbf{B}_{gm}\mathbf{Z}/2\mathbf{Z} \to \mathbf{P}^\infty$.

For any k-linear representation V of $\mathbf{Z}/2\mathbf{Z}$, one may attach a "vector bundle on $\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2\mathbf{Z}$". On $\{\pm 1\}\backslash(\mathbf{A}^n - \{0\})$, it is $\{\pm 1\}\backslash(\mathbf{A}^n - \{0\} \times V)$ as we did before in the case of a permutation representation. We have a short exact sequence of representations of $\mathbf{Z}/2\mathbf{Z}$:

$$0 \to k \xrightarrow{+} (k^2, \tau) \xrightarrow{-} \chi \to 0 \, .$$

where τ inverts the two factors and χ is the nontrivial (selfdual) character of $\mathbf{Z}/2\mathbf{Z}$. To this exact sequence is attached the exact sequence of vector bundles:

$$0 \to \mathcal{O} \to \xi \to \xi/\mathcal{O} \to 0 \, .$$

Then $\xi/\mathcal{O}$ is attached to the character χ. In terms of the $\mathbf{G}_{\mathrm{m}}$-torsors associated to $\xi/\mathcal{O}$ and the inverse image of $\mathcal{O}(-1)$, the result follows from the isomorphism $\{\pm 1\}\backslash((\mathbf{A}^n - \{0\}) \times \mathbf{G}_{\mathrm{m}}) \xrightarrow{\sim} (\{\pm 1\}\backslash(\mathbf{A}^n - \{0\})) \times_{\mathbf{P}^{n-1}} (\mathbf{A}^n - \{0\})$ that maps the class of $[v, \lambda]$ to $([v], \lambda v)$.

We proved this:

Proposition

We let $t \in \widetilde{H}^{2,1}(S^{2,1})$ $(S^{2,1} \simeq \mathbf{A}^1/(\mathbf{A}^1 - \{0\}))$ be the tautological class. Then, for all $i \geq 0$ and $x \in \widetilde{H}^{2d,d}(\mathcal{X})$, $P^i(x \cup t) = P^i(x) \cup t$ and $B^i(x \cup t) = B^i(x) \cup t$.

This shows that the definition we gave of the operations P^i and B^i on $\widetilde{H}^{2d,d}$ are compatible for different d. We have thus defined *stable* cohomology operations for all $i \geq 0$:

$$P^i : \widetilde{H}^{p,q}(\mathcal{X}) \to \widetilde{H}^{p+2i,q+i}(\mathcal{X})$$

$$B^i : \widetilde{H}^{p,q}(\mathcal{X}) \to \widetilde{H}^{p+2i+1,q+i}(\mathcal{X})$$

for all $(p,q) \in \mathbf{Z}$ and $\mathcal{X} \in \mathcal{H}_\bullet(k)$. It follows that these operations are additive. (We also know that $B^i = \beta P^i$, i.e., $\mathrm{Sq}^{2j+1} = \beta\,\mathrm{Sq}^{2j}$.)

Proposition

$P^0 = \mathrm{Sq}^0$ is the identity and $B^0 = \mathrm{Sq}^1 = \beta$.

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We know that on $\widetilde{H}^{2d,d}$, P^0 is the multiplication by some $c_d \in \mathbf{Z}/2\mathbf{Z}$. The fact that P^0 is a stable operation show that $c_d = c_0$. For obvious reasons, $c_0 = 1$ (using the formula $P(t) = t \cup v$, one may also observe that $c_1 = 1$). It follows that P^0 is the identity. Then, $B^0 = \beta P^0 = \beta$.

Proposition

If $x \in \widetilde{H}^{\star,\star}(\mathscr{X})$ and $y \in \widetilde{H}^{\star,\star}(\mathscr{Y})$, we have:

$$P^k(x \cup y) = \sum_{i+j=k} P^i(x) \cup P^j(y) + \tau \sum_{i+j=k-1} B^i(x) \cup B^j(y)$$

$$B^k(x \cup y) = \sum_{i+j=k} P^i(x) \cup B^j(y) + \sum_{i+j=k} B^i(x) \cup P^j(y) + \rho \sum_{i+j=k-1} B^i(x) \cup B^j(y)$$

Proposition

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$$B^k(x \cup y) = \sum_{i+j=k} P^i(x) \cup B^j(y) + \sum_{i+j=k} B^i(x) \cup P^j(y) + \rho \sum_{i+j=k-1} B^i(x) \cup B^j(y)$$

One may assume $x \in \widetilde{H}^{2d,d}(\mathcal{X})$ and $y \in \widetilde{H}^{2d',d'}(\mathcal{Y})$. Then:

$$P(x) \cdot P(y) = (\textstyle\sum_{i=0}^{d} P^i(x) v^{d-i} + \sum_{i=0}^{d-1} B^i(x) u v^{d-i-1}) \cdot (\sum_{j=0}^{d'} P^j(y) v^{d'-j} + \sum_{j=0}^{d-1} B^j(x) u v^{d-j-1})$$

Then, one uses the computation $u^2 = \tau v + \rho u$ and the identification with:

$$P(xy) = \sum_{k=0}^{d+d'} P^k(xy) v^{d+d'-k} + \sum_{k=0}^{d+d'-1} B^k(xy) u v^{d+d'-1-k}$$

Proposition

If $x \in \widetilde{H}^{2d,d}(\mathcal{X})$, then $P^d(x) = x^2$.

We use the following lemma for $i = d$, $n = 2$, $U = \mathbf{A}^? - \{0\}$ and $G = \{\pm 1\}$:

Lemma

The composition

$$K_i \to K_i \wedge (G \backslash U)_+ \xrightarrow{P} K_{in}$$

where the first map is induced by a rational point of U is the "raising to the power n" morphism.

The restriction map $\widetilde{H}^{\star,\star}(\mathcal{X} \wedge (\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2)_+) \to \widetilde{H}^{\star,\star}(\mathcal{X})$ sends $P(x)$ to x^2. Moreover, the images of u and v vanish, to that $P(x)$ is also sent to $P^d(x)$.

Corollary

If $x \in \widetilde{H}^{p,q}(\mathcal{X})$ with $d \geq q$ and $d > p - q$, then $P^d(x) = 0$.

Proof.

Using suspensions with S^1 or $\mathbf{G}_{\mathrm{m}}$, one may assume $x \in \widetilde{H}^{2d-1,d}(\mathcal{X})$. Let $\tilde{x} = s \wedge x \in \widetilde{H}^{2d,d}(S^1 \wedge \mathcal{X})$ where $s \in H^{1,0}(S^1)$ is the tautological class. We have to show that $\tilde{x}^2 = 0$. This class is induced by a morphism in $\mathcal{H}_\bullet(k)$ that factors through the diagonal:

$$S^1 \wedge \mathcal{X} \to S^2 \wedge \mathcal{X}^{\wedge 2}$$

which is the $\wedge$-product of two morphisms, but the first one $S^1 \to S^2$ is the zero map because the Riemann sphere is simply connected. □

Proposition

Let $X \in Sm/k$. Let L be a line bundle on X. Let $c_1(L) \in H^{2,1}(X)$ be its first Chern class.
Then,

$$P(c_1(L)) = c_1(L)^2 + c_1(L)v$$

In other words,

$$P^0(c_1(L)) = c_1(L) \quad P^1(c_1(L)) = c_1(L)^2 \quad B^0(c_1(L)) = 0$$

This follows from the preceding results for P^0, P^1 and B^0.

Corollary

Let $X \in Sm/k$. The sub-$\mathbf{F}_2$-algebra of $H^{2\star,\star}(X) = CH^\star(X)/2$ generated by Chern classes of vector bundles on X is stable under the operations P^n and killed by the operations B^n.

Corollary

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It is true for 1 and first Chern classes of line bundles.
Consider the vector bundle $V = L_1 \oplus \cdots \oplus L_d$ on $(\mathbf{P}^k)^d$ (for k big enough) where L_i is the inverse image of $\mathscr{O}(1)$ by the ith projection on $\mathbf{P}^k$. Define $x_i = c_1(L_i)$. $c_k(V)$ identifies to a symmetric polynomial involving the d variables $x_1, \ldots, x_d$. Using the previous formulas, $P^n(c_k(V))$ may also be identified with a symmetric polynomial involving $x_1, \ldots, x_d$. Then, there exists a polynomial $f \in \mathbf{F}_2[c_1, \ldots, c_d]$ such that

$$P^n(c_k(V)) = f(c_1(V), \ldots, c_d(V))$$

Standard arguments shows that if this is true for this specific V on $(\mathbf{P}^k)^d$ (which is true by definition), then it is true for all bundles of rank d on schemes in Sm/k.

We use the identification
$\widetilde{H}^{\star,\star}(\mathcal{X} \wedge (\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2\mathbf{Z})_+) \simeq \widetilde{H}^{\star,\star}(\mathcal{X}) \otimes_{H^{\star,\star}(k)} H^{\star,\star}(\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2\mathbf{Z})$:

Corollary

$P(v) = v^2 \otimes 1 + v \otimes v$ and $P(u) = u \otimes v + v \otimes v$.

(The second formula does not make sense as it is. If $x \in \widetilde{H}^{p,q}(\mathcal{X})$ with $p \leq 2q$, one may identify x to a class $\tilde{x} \in \widetilde{H}^{2q,q}(S^{2q-p} \wedge \mathcal{X})$. Then, $P(\tilde{x})$ makes sense, and we define $P(x) \in \widetilde{H}^{\star,\star}(\mathcal{X} \wedge (\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2\mathbf{Z})_+)$ from $P(\tilde{x})$ by using the suspension isomorphism in the opposite direction.)

The computation of $P(v)$ follows from the formula for $P(c_1(L))$ and the identity $v = c_1(\mathscr{O}(1))$.
We may write $P(u)$ as:

$$P(u) = P^0(u) \otimes v + P^1(u) \otimes 1 + \beta u \otimes u = u \otimes v + v \otimes u$$

because $P^1(u) = 0$.

Proposition

For all $i, k \geq 0$, the following relations hold in $H^{\star,\star}(\mathbf{P}^\infty) \subset H^{\star,\star}(\mathbf{B}_{gm}\mathbf{Z}/2\mathbf{Z})$:

$$P^i(v^k) = \binom{k}{i} v^{k+i}, \quad B^i(v^k) = 0$$

for all $i, k \geq 0$. In $H^{\star,\star}(\mathbf{B}_{gm}\mathbf{Z}/2\mathbf{Z})$, we have:

$$P^i(uv^k) = \binom{k}{i} uv^{k+i}, \quad B^i(uv^k) = \binom{k}{i} v^{k+i+1}$$

Proof.

The first series of identities follows from:

$$P(v^k) = P(v)^k = (v^2 \otimes v + v \otimes v)^k = \sum_{i=0}^{k} \binom{k}{i} v^{k+i} \otimes v^{k-i} = \sum_{i=0}^{k} P^i(v^k) v^{k-i}$$

The other series come from the multiplication formulas. □

We defined $P\colon \widetilde{H}^{2d,d}(\mathcal{X}) \to \widetilde{H}^{4d,2d}(\mathcal{X} \wedge \mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2\mathbf{Z}_+)$. One may iterate it so as to obtain a map:

$$P \circ P\colon \widetilde{H}^{2d,d} \to \widetilde{H}^{8d,4d}(\mathcal{X} \wedge (\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2\mathbf{Z} \times \mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2\mathbf{Z})_+)$$

One may identify the target group as a bigraded component of

$$\widetilde{H}^{\star,\star}(\mathcal{X}) \otimes_{H^{\star,\star}(k)} H^{\star,\star}(\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2\mathbf{Z}) \otimes_{H^{\star,\star}(k)} H^{\star,\star}(\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2\mathbf{Z})$$

Theorem

Let $x \in \widetilde{H}^{2d,d}(\mathcal{X})$. Then, $(P \circ P)(x)$ is invariant under the exchange of the two copies of $H^{\star,\star}(\mathbf{B}_{gm}\mathbf{Z}/2\mathbf{Z})$ in the tensor product.

Theorem

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The sketch of proof is that $P \circ P$ can be identified with the construction P for the action of $G = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ on $\{1,2\} \times \{1,2\}$ $(n = 4)$. This action can be extended to an action of the semidirect product $G \rtimes \mathbf{Z}/2\mathbf{Z}$ where $\mathbf{Z}/2\mathbf{Z}$ acts on G and $\{1,2\} \times \{1,2\}$ by permutation of the two factors. Then, we can apply the construction P to this action of $G \rtimes \mathbf{Z}/2\mathbf{Z}$ which refines the class $(P \circ P)(x)$ and look at the commutative diagram:

$$\begin{array}{ccc}
\widetilde{H}^{\star,\star}(\mathcal{X} \wedge \mathbf{B}_{\mathrm{gm}}(G \rtimes \mathbf{Z}/2)_+) & \xrightarrow{\text{res}} & \widetilde{H}^{\star,\star}(\mathcal{X} \wedge \mathbf{B}_{\mathrm{gm}} G_+) \\
\Big\downarrow \text{interior automorphism}\sim\mathrm{Id} & & \Big\downarrow \text{switch of two factors } \mathbf{Z}/2\mathbf{Z} \\
\widetilde{H}^{\star,\star}(\mathcal{X} \wedge \mathbf{B}_{\mathrm{gm}}(G \rtimes \mathbf{Z}/2)_+) & \xrightarrow{\text{res}} & \widetilde{H}^{\star,\star}(\mathcal{X} \wedge \mathbf{B}_{\mathrm{gm}} G_+)
\end{array}$$

Corollary (Adem relations)

Assume a and b are integers satisfying $0 < a < 2b$. If a is even and b odd,

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j + \sum_{\substack{j=1 \\ odd}}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \rho \, \mathrm{Sq}^{a+b-j-1} \mathrm{Sq}^j$$

$$\text{If } a \text{ and } b \text{ are odd,} \quad \mathrm{Sq}^a \mathrm{Sq}^b = \sum_{\substack{j=0 \\ odd}}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j$$

$$\text{If } a \text{ and } b \text{ are even,} \quad \mathrm{Sq}^a \mathrm{Sq}^b = \sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \tau^{j \bmod 2} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j$$

If a is odd and b is even,

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{\substack{j=0 \\ even}}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j + \sum_{\substack{j=1 \\ odd}}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-1-2j} \rho \, \mathrm{Sq}^{a+b-j-1} \mathrm{Sq}^j$$

Some remarks:

- All monomials in the right member are of the form $\mathrm{Sq}^i\,\mathrm{Sq}^j$ with $i \geq 2j$.
- The first equation implies the second by applying β.
- Similarly, the third implies the fourth.
- If $\rho = 0$ (i.e., -1 is a square in k, for instance if $k = \mathbf{C}$), then we get exactly the same formulas as in topology (through the identification $\tau = 1$) where they reduce to: $\mathrm{Sq}^a\,\mathrm{Sq}^b = \sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j}\,\mathrm{Sq}^j$.
- If $\rho \neq 0$, the formulas are a little bit more complicated.

Here are some details about the proof of the "corollary". We have $P(P(x)) = \sum_{j=0}^{2d} P^j(P(x)) \otimes v^{2d-j} + \sum_{j=0}^{2d-1} B^j(P(x)) \otimes uv^{2d-1-j}$ and $P(x) = \sum_{i=0}^{d} P^i(x)v^{d-i} + \sum_{i=0}^{d-1} B^i(x)uv^{d-1-i}$. Using previous formulas, we get:

$$\begin{aligned}
P(P(x)) &= \sum_{j=0}^{2d}\sum_{i=0}^{d}\sum_{k=0}^{j}\binom{d-i}{j-k} P^kP^i(x)\otimes v^{d+j-k-i}\otimes v^{2d-j} \\
&+ \sum_{j=0}^{2d-1}\sum_{i=0}^{d}\sum_{k=0}^{j}\binom{d-i}{j-k} B^kP^i(x)\otimes v^{d+j-k-i}\otimes uv^{2d-1-j} \\
&+ \sum_{j=0}^{2d}\sum_{i=0}^{d-1}\sum_{k=0}^{j}\binom{d-1-i}{j-k} P^kB^i(x)\otimes uv^{d+j-k-i-1}\otimes v^{2d-j} \\
&+ \tau\sum_{j=0}^{2d}\sum_{i=0}^{d-1}\sum_{k=0}^{j-1}\binom{d-1-i}{j-1-k} B^kB^i(x)\otimes v^{d+j-k-i-1}\otimes v^{2d-j} \\
&+ \sum_{j=0}^{2d-1}\sum_{i=0}^{d-1} B^j(B^i(x)uv^{d-1-i})\otimes uv^{2d-1-j}
\end{aligned}$$

$$
\begin{aligned}
P(P(x)) &= \sum_{j=0}^{2d}\sum_{i=0}^{d}\sum_{k=0}^{j}\binom{d-i}{j-k}P^kP^i(x)\otimes v^{d+j-k-i}\otimes v^{2d-j}\\
&+ \sum_{j=0}^{2d-1}\sum_{i=0}^{d}\sum_{k=0}^{j}\binom{d-i}{j-k}B^kP^i(x)\otimes v^{d+j-k-i}\otimes uv^{2d-1-j}\\
&+ \sum_{j=0}^{2d}\sum_{i=0}^{d-1}\sum_{k=0}^{j}\binom{d-1-i}{j-k}P^kB^i(x)\otimes uv^{d+j-k-i-1}\otimes v^{2d-j}\\
&+ \tau\sum_{j=0}^{2d}\sum_{i=0}^{d-1}\sum_{k=0}^{j-1}\binom{d-1-i}{j-1-k}B^kB^i(x)\otimes v^{d+j-k-i-1}\otimes v^{2d-j}\\
&+ \sum_{j=0}^{2d-1}\sum_{i=0}^{d-1}\sum_{k=0}^{j}\binom{d-1-i}{j-k}P^kB^i(x)\otimes v^{d+j-k-i}\otimes uv^{2d-1-j}\\
&+ \sum_{j=0}^{2d-1}\sum_{i=0}^{d-1}\sum_{k=0}^{j}\binom{d-1-i}{j-k}B^kB^i(x)\otimes uv^{d+j-k-i-1}\otimes uv^{2d-1-j}\\
&+ \rho\sum_{j=0}^{2d-1}\sum_{i=0}^{d-1}\sum_{k=0}^{j-1}\binom{d-1-i}{j-1-k}B^kB^i(x)\otimes v^{d+j-k-i-1}\otimes uv^{2d-1-j}
\end{aligned}
$$

Let $p, q \geq 0$. The coefficient of $uv^p \otimes v^q$ in $P(P(x))$ is:

$$\alpha_{p,q} = \sum_{i=0}^{d-1} \binom{d-i-1}{p-(d-i-1)} P^{3d-p-q-i-1} B^i(x)$$

It must be the same as the coefficient of $v^q \otimes uv^p$:

$$\begin{aligned} \beta_{p,q} &= \sum_{i=0}^{d-1} \binom{d-i}{q-(d-i)} B^{3d-p-q-i-1} P^i(x) \\ &+ \sum_{i=0}^{d-1} \binom{d-1-i}{q-(d-i)} P^{3d-p-q-i-1} B^i(x) \\ &+ \rho \sum_{i=0}^{d-1} \binom{d-1-i}{q-(d-i)} B^{3d-p-q-i-2} B^i(x) \end{aligned}$$

Assume $a = 2a'$ and $b = 2b' + 1$ are such that $0 < a < 2b$ (i.e., $a' \leq 2b'$). We would like a formula for

$$\alpha_{p,q} = \sum_{i=0}^{d-1} \binom{d-i-1}{p-(d-i-1)} P^{3d-p-q-i-1} B^i(x)$$

We fix $s \geq 0$ and set $p = 2^s - 1$, $d = 2^s + b'$, $q = 2^{s+1} + 2b' - a'$.

Lemma

Then, $\alpha_{p,q} = P^{a'} B^{b'}(x) = \mathrm{Sq}^a \mathrm{Sq}^b(x)$.

This expression $P^{a'} B^{b'}$ is the term corresponding to $i = b'$ (because $p = d - b' - 1$), we have to show the other coefficients are zero. For obvious reasons, the coefficient $\binom{d-i-1}{p-(d-i-1)} = 0$ if $i < b'$. We shall show that for this specific choice of p, this is even if $i > b'$ also.
Introducing $\delta = p - (d - i - 1)$, we have to show that $\binom{p-\delta}{\delta} \equiv 0 \mod 2$ if $0 < \delta \leq \frac{p}{2}$.

Lemma

Assume $i, j \geq 0$, then $\binom{i+j}{i} \equiv 1 \mod 2$ if and only if there is no carry when computing the sum $i + j$ in the binary numeral system.

It follows from the computation of the 2-adic valuation of $n!$:

$$v_2(n!) = \sum_{k \geq 1} \left\lfloor \frac{n}{2^k} \right\rfloor$$

We may also say that if $i, j \geq 0$, $\binom{i}{j} \equiv 1 \mod 2$ if and only if there is no carry when computing $i - j$ in $\mathbf{Z}_2$ (includes the case $j > i$...).

For instance, it follows from the lemma that $\binom{i}{j} \equiv \binom{2i}{2j} \mod 2$.

Assume $p = 2^s - 1$ and $0 < \delta \leq \frac{p}{2}$. To compute the parity of $\binom{p-\delta}{\delta}$, we want to look at possible carry when doing the difference $(p - \delta) - \delta$.
But, all the digits of p are 1. Then, for any nonzero digit of δ, the corresponding digit of $p - \delta$ is zero. This shows that a carry will occur, so that $\binom{p-\delta}{\delta} \equiv 0 \mod 2$.

We come back to $\beta_{p,q} = \alpha_{p,q}$.

$$\begin{aligned}\beta_{p,q} &= \sum_{i=0}^{d-1} \binom{d-i}{q-(d-i)} B^{3d-p-q-i-1} P^i(x) \\ &+ \sum_{i=0}^{d-1} \binom{d-1-i}{q-(d-i)} P^{3d-p-q-i-1} B^i(x) \\ &+ \rho \sum_{i=0}^{d-1} \binom{d-1-i}{q-(d-i)} B^{3d-p-q-i-2} B^i(x)\end{aligned}$$

In the first sum, it suffices to take into account those i such that $q-(d-i) \leq d-i$, i.e, $2i \leq 2d-q = a' = \frac{a}{2}$, then:

$$\binom{d-i}{q-(d-i)} = \binom{d-i}{2d-2i-q} \equiv \binom{2d-2i}{4d-4i-2q} = \binom{2^{s+1}+b-1-2i}{a-4i}$$

Given a and b, for s big enough, this is $\equiv \binom{b-1-2i}{a-4i}$.

Using the correspondence $j = 2i$, we showed that

$$\sum_{i=0}^{d-1} \binom{d-i}{q-(d-i)} B^{3d-p-q-i-1} P^i(x) = \sum_{\substack{j=0 \\ \text{even}}}^{\frac{a}{2}} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j(x)$$

Similarly, with $j = 2i + 1$,

$$\sum_{i=0}^{d-1} \binom{d-1-i}{q-(d-i)} P^{3d-p-q-i-1} B^i(x) = \sum_{\substack{j=0 \\ \text{odd}}}^{\frac{a}{2}} \binom{b-2-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j(x)$$

Then, one may believe that there is a mistake, but when j is odd, we have:

$$\binom{b-1-j}{a-2j} = \binom{b-2-j}{a-2j} + \binom{b-2-j}{a-2j-1} \equiv \binom{b-2-j}{a-2j} \mod 2$$

because $b - 2 - j$ is even and $a - 2j - 1$ is odd.

Finally, we get:

$$\beta_{p,q} = \sum_{j=0}^{\frac{a}{2}} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j}\,\mathrm{Sq}^{j}(x) + \rho \sum_{\substack{j=0 \\ \text{odd}}}^{\frac{a}{2}} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j-1}\,\mathrm{Sq}^{j}(x)$$

This equals $\alpha_{p,q} = \mathrm{Sq}^a\,\mathrm{Sq}^b(x)$.

This shows the first expected relation for $x \in \widetilde{H}^{2d,d}(\mathcal{X})$ for d of the form $2^s + b'$ and s big enough, which is sufficient using suspensions.

This third relation is similar but uses a combination of two different equalities of coefficients of $P(P(x))$.

Definition

Let I be a sequence of integers $(\varepsilon_0, r_1, \varepsilon_1, r_2, \dots)$ that is ultimately zero and such that $\varepsilon_i \in \{0,1\}$. We define:

$$P^I = \beta^{\varepsilon_0} P^{s_1} \beta^{\varepsilon_1} P^{s_2} \dots$$

where $s_i = \sum_{k \geq i} (\varepsilon_k + r_k) 2^{k-i}$ (note that $s_i \geq 2s_{i+1} + \varepsilon_i$). These elements are called "admissible monomials".

Definition (Steenrod algebra)

We denote $H^{\star,\star} = H^{\star,\star}(k)$. This algebra acts by multiplication on motivic cohomology: then any element in $H^{\star,\star}$ defines a stable cohomology operation. We denote $A^{\star,\star}$ the algebra of stable cohomology operations generated by $H^{\star,\star}$, β and P^n $(n \geq 1)$.

We consider $A^{\star,\star}$ as a (left-)module over $H^{\star,\star}$.

Proposition

$A^{\star,\star}$ is a free $H^{\star,\star}$-module with a basis consisting of the admissible monomials.

Relations obtained until now shows that the module generated by the admissible monomials P^I is an algebra. The proof that they constitute a basis is similar to the topological situation:

"One may detect a nontrivial linear combination $\sum_I a_I P^I$ by looking at its action on $H^{\star,\star}((\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2)^n)$ for a big enough n."

Definition

We denote $A_{\star,\star}$ the $H^{\star,\star}$-module dual to $A^{\star,\star}$. The component $A_{p,q}$ maps $A^{i,j}$ into $H^{i-p,j-q}$.
This $H^{\star,\star}$-module is free with a basis given by elements $\theta(I)^{\star}$ dual of the basis of admissible monomials P^{I}.

The fact that we are in bigraded situation (and the distribution of bidegrees) implies that these modules behaves as if they were free of finite type.

For $C \in A^{\star,\star}$ and $\alpha \in A_{\star,\star}$, the element $\alpha(C) \in H^{\star,\star}$ is denoted $\langle \alpha, C \rangle$.

Definition

Let $X \in Sm/k$. We define

$$\lambda\colon H^{\star,\star}(X) \to A_{\star,\star} \otimes_{H^{\star,\star}} H^{\star,\star}(X)$$

the unique map (additive but not $H^{\star,\star}$-linear) such that for any $x \in H^{\star,\star}(X)$, if $\lambda(x) = \sum_i \alpha_i \otimes y_i$, then, for any $C \in A^{\star,\star}$, we have:

$$C(x) = \sum_i \langle \alpha_i, C \rangle \, y_i$$

(Note that $\lambda(x) = \sum_I \theta(I)^\star \otimes P^I(x)$.)

Then, $\lambda(x) \in A_{\star,\star} \otimes_{H^{\star,\star}} H^{\star,\star}(X)$ reflects the action of $A^{\star,\star}$ on this class x.

Definition

For $k \geq 0$, we define $\xi_k \in A_{2^{k+1}-2,2^k-1}$ (resp. $\tau_k \in A_{2^{k+1}-1,2^k-1}$) as those of the $\theta(I)^\star$ that are dual to the admissible monomials $M_k = P^{2^{k-1}} \dots P^2 P^1 \in A^{\star,\star}$ (resp. $M_k\beta$).

Proposition

For "$X = \mathbf{B}_{gm}\mathbf{Z}/2\mathbf{Z}$", we have:

$$\lambda(v) = \sum_{k=0}^{\infty} \xi_k \otimes v^{2^k} \qquad \lambda(u) = \xi_0 \otimes u + \sum_{k=0}^{\infty} \tau_k \otimes v^{2^k}$$

Here, X is not in Sm/k, but is a colimit of such. In this particular case, it makes sense to define $\lambda(u)$ or $\lambda(v)$ as series.

To show that $\lambda(v) = \sum_{k=0}^{\infty} \xi_k \otimes v^{2^k}$, we have to show that the only (admissible or not!) monomials N involving the P^n and β such that $N(v) \neq 0$ are the monomials $M_k = P^{2^{k-1}} \dots P^2 P^1$, $k \geq 0$ and that $M_k(v) = v^{2^k}$.

We have $P^1(v) = v^2 \in H^{4,2}(\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2)$, $P^2P^1(v) = P^2(v^2) = v^4$, etc. A simple induction shows that $M_k(v) = v^{2^k}$.
Assume that a monomial $N = \beta N'$ or $N = P^n N'$ $(n > 0)$ is such that $N(v) \neq 0$. Then, $N'(v) \neq 0$. By induction, we must have $N' = M_k$ for some $k \geq 0$. We have, $M_k(v) = v^{2^k}$. Then, $\beta M_k(v) = 0$. For degree reasons, $P^n M_k(v) = 0$ if $n > 2^k$. If $0 < n < 2^k$, we have

$$N(v) = P^n(v^{2^k}) = \binom{2^k}{n} v^{2^k+n} = 0$$

Then, we must have $n = 2^k$, and $N = M_{k+1}$.
For u, N can be the empty word, which corresponds to the identity $P^0 = M_0$.
Otherwise, the last letter must be β, and the previous argumentation shows that $N = M_k\beta$.

Let us have a look at these formulas again:

Proposition

$$\lambda(v) = \sum_{k=0}^{\infty} \xi_k \otimes v^{2^k} \qquad \lambda(u) = \xi_0 \otimes u + \sum_{k=0}^{\infty} \tau_k \otimes v^{2^k}$$

Definition

We define a comultiplication:

$$\Psi^\star \colon A^{\star,\star} \to A^{\star,\star} \otimes_{H^{\star,\star}} A^{\star,\star}$$

(both copies of $A^{\star,\star}$ are equipped with the left-module structure.) in such a way that for any $C \in A^{\star,\star}$, $\Psi^\star(C) = \sum_i D_i \otimes E_i$ is the unique element such that for all motivic cohomology classes x and y:

$$C(xy) = \sum_i D_i(x)E_i(y)$$

$\Psi^\star$ is co-associative, cocommutative (this reflects associativity and commutativity of the multiplication of cohomology classes) and $H^{\star,\star}$-linear.

Uniqueness of $\Psi^\star(C)$ is deduced from the fact that "$A^{\star,\star}$ acts faithfully on $H^{\star,\star}(\mathbf{B}_{\mathrm{gm}}\mathbf{Z}/2\mathbf{Z}^{\mathrm{high}})$".

For the existence, we use the following lemmas:

Lemma

$$\Psi^\star \beta = \beta \otimes \mathrm{Id} + \mathrm{Id} \otimes \beta$$

$$\Psi^\star P^n = \sum_{i+j=n} P^i \otimes P^j + \tau \sum_{i+j=n-1} B^i \otimes B^j$$

Lemma

If $\Psi^\star(C) = \sum_i A_i \otimes B_i$ *and* $\Psi^\star(D) = \sum_j E_j \otimes F_j$, *then*

$$\Psi^\star(CD) = \sum_{i,j} A_i E_j \otimes B_i F_j$$

Lemma

In $A_{\star,\star} \simeq A_{\star,\star} \otimes_{H^{\star,\star}} H^{\star,\star}$, we have:

$$\lambda(1) = \xi_0$$

This also means that $\langle \xi_0, C \rangle = C(1)$ for all $C \in H^{\star,\star}$. This follows from the fact that 1 is killed by all monomials excepted Id.

Lemma

$\xi_0 \colon A^{\star,\star} \to H^{\star,\star}$ is the coünit of $\Psi^\star$, i.e., the composition:

$$A^{\star,\star} \xrightarrow{\Psi^\star} A^{\star,\star} \otimes_{H^{\star,\star}} A^{\star,\star} \xrightarrow{\mathrm{Id} \otimes \xi_0} A^{\star,\star} \otimes_{H^{\star,\star}} H^{\star,\star} \xrightarrow{\simeq} A^{\star,\star}$$

is the identity.

We shall dualisize the comultiplication $\Psi^\star$ on $A^{\star,\star}$.

We define a $H^{\star,\star}$-bilinear pairing $\langle \alpha \otimes \beta, C \otimes D \rangle = \langle \alpha, C \rangle \cdot \langle \beta, D \rangle$ on $(A_{\star,\star} \otimes_{H^{\star,\star}} A_{\star,\star}) \times (A^{\star,\star} \otimes_{H^{\star,\star}} A^{\star,\star})$.

Definition

We define a product law on $A_{\star,\star}$. It is characterized by the relation:

$$\langle \alpha\beta, C \rangle = \langle \alpha \otimes \beta, \Psi^\star C \rangle$$

for $\alpha, \beta \in A_{\star,\star}$ and $C \in A^{\star,\star}$.

Proposition

$A_{\star,\star}$ is a commutative $H^{\star,\star}$-algebra. Its unit is ξ_0.
For any $X \in Sm/k$, the map

$$\lambda \colon H^{\star,\star}(X) \to A_{\star,\star} \otimes_{H^{\star,\star}} H^{\star,\star}(X)$$

is a morphism of $H^{\star,\star}$-algebras.

Proposition

Let $C \in A^{\star,\star}$. *Then:* $C(v^{2^j}) = \sum_{i \geq 0} \left\langle \xi_i^{2^j}, C \right\rangle v^{2^{i+j}}$

It is equivalent to saying that:

$$\lambda(v^{2^j}) = \sum_{i \geq 0} \xi_i^{2^j} \otimes v^{2^{i+j}}$$

We already know the case $j = 0$:

$$\lambda(v) = \sum_{i \geq 0} \xi_i \otimes v^{2^i}$$

Then, we use $\lambda(v^{2^j}) = \lambda(v)^{2^j}$.

Theorem

The ring $A_{\star,\star}$ is the commutative $H^{\star,\star}$-algebra generated by elements $\tau_k \in A_{2^{k+1}-1,2^k-1}$ $(k \geq 0)$ and $\xi_k \in A_{2^{k+1}-2,2^k-1}$ $(k \geq 1)$ subjected to the following relations for all $k \geq 0$:

$$\tau_k^2 = (\tau + \rho\tau_0)\xi_{k+1} + \rho\tau_{k+1}$$

The relations follows from the analysis of the coefficient of $v^{2^{k+1}}$ in:

$$\lambda(u)^2 = \lambda(u^2) = \lambda(\tau)\lambda(v) + \lambda(\rho)\lambda(u)$$

and the identities $\lambda(\tau) = \tau + \rho\tau_0$ and $\lambda(\rho) = \rho$. Remember that:

$$\lambda(v) = \sum_{k=0}^{\infty} \xi_k \otimes v^{2^k} \qquad \lambda(u) = \xi_0 \otimes u + \sum_{k=0}^{\infty} \tau_k \otimes v^{2^k}$$

To prove the theorem, we have to show that the elements

$$\omega(I) = \prod_{k\geq 0} \tau_k^{\varepsilon_k} \prod_{k\geq 1} \xi_k^{r_k} \in A_{\star,\star}$$

for sequences $I = (\varepsilon_0, r_1, \varepsilon_1, \dots)$ as above constitute a basis of $A_{\star,\star}$ as a $H^{\star,\star}$-module.

Lemma

We use the lexicographic order (starting from the right) on such sequences I. Then $\langle \omega(I), P^I \rangle = 1$ and for $I < J$, $\langle \omega(J), P^I \rangle = 0$.

Then, matrix $\langle \omega(I), P^J \rangle$ of the coefficients of the $\omega(I)$ in the basis on the $\theta(J)^\star$ is upper triangular with 1 in the diagonal.

When proving that the $\omega(I)$ generate $A_{\star,\star}$, one uses the fact that for a fixed bidegree (p, q), there exists only finitely many J such that there exists $x \neq 0 \in H^{i,j}$ (we use the bound $i \leq j$) such that the bidegree of $x\theta(J)^\star$ is (p, q).

Denote $J = (\tilde{\varepsilon}_0, \tilde{r}_1, \dots)$. We do an induction on the total degree of $\omega(J)$ to show that $\langle \omega(J), P^I \rangle = 0$ if $I < J$.
Assume that the last nonzero coefficient of J is $\tilde{r}_k \neq 0$. Introduce J' such that $\omega(J) = \omega(J')\xi_k$:

$$\left\langle \omega(J), P^I \right\rangle = \left\langle \omega(J') \otimes \xi_k, \Psi^\star(P^I) \right\rangle$$

Expand $\Psi^\star(P^I)$ as a sum of $C \otimes D$ where D is a monomial involving β or P^i:

$$\left\langle \omega(J') \otimes \xi_k, C \otimes D \right\rangle = \left\langle \omega(J'), C \right\rangle \left\langle \xi_k, D \right\rangle$$

If this is nonzero, we must have $D = M_k = P^{2^{k-1}} \dots P^2 P^1$.
As $I < J$, I is of the form $I = (\varepsilon_0, r_1, \varepsilon_1, \dots, \varepsilon_{k-1}, r_k, 0, \dots)$.
We know how to expand $\Psi^\star P^I$, where $P^I = \beta^{\varepsilon_0} P^{s_1} \beta^{\varepsilon_1} \dots P^{s_k}$. Basically, $\Psi^\star P^{s_{k-j}} = P^{s_{k-j} - 2^j} \otimes P^{2^j} + \text{other terms}$.
We see there shall be a term $C \otimes M_k$ only if $r_k \geq 1$. Then, $C = P^{I'}$ with $I' = (\varepsilon_0, r_1, \varepsilon_1, \dots, \varepsilon_{k-1}, r_k - 1, 0, \dots)$, then:

$$\left\langle \omega(J), P^I \right\rangle = \left\langle \omega(J'), P^{I'} \right\rangle = 0 \text{ by induction}$$

Similar arguments for the case when the last coefficient of J is a $\tilde{\varepsilon}_?$ and for $\langle \omega(I), P^I \rangle$.

- $A^{\star,\star}$ has a right-module structure over $H^{\star,\star}$: it is $H^{\star,\star}$-bimodule-$H^{\star,\star}$.
- $A_{\star,\star}$ is $H^{\star,\star}$-bimodule-$H^{\star,\star}$.

Lemma

If $\alpha \in A_{\star,\star}$ and $x \in H^{\star,\star}$, $\alpha.x = \lambda(x)\alpha$.

For all $C \in A^{\star,\star}$, we have to check:

$$\langle \alpha.x, C\rangle = \langle \alpha, Cx\rangle = \langle \lambda(x)\alpha, C\rangle$$

Assume $\Psi^\star C = \sum_i D_i \otimes E_i$. Then, $Cx = \sum_i D_i(x) \cdot E_i \in A^{\star,\star}$.

$$\begin{aligned} \langle \lambda(x)\alpha, C\rangle &= \sum_i \langle \lambda(x) \otimes \alpha, D_i \otimes E_i\rangle = \sum_i D_i(x) \langle \alpha, E_i\rangle \\ &= \left\langle \alpha, \sum_i D_i(x) \cdot E_i \right\rangle = \langle \alpha, Cx\rangle \end{aligned}$$

Note that the two structures of modules on $A_{\star,\star}$ are induced by the ring morphisms $H^{\star,\star} \to A_{\star,\star}$: $x \longmapsto x\xi_0$ and $x \longmapsto \lambda(x)$.

We introduce $A^{\star,\star} \otimes_{r,H^{\star,\star},l} A^{\star,\star}$ as a left-$H^{\star,\star}$-module. This comes from the $H^{\star,\star}$-bimodule structure on the first $A^{\star,\star}$ and the left-module structure on the second.

Lemma

Tensor products $P^I \otimes P^J$ of admissible monomials give a basis of $A^{\star,\star} \otimes_{r,H^{\star,\star},l} A^{\star,\star}$ as a left-$H^{\star,\star}$-module.
Similarly, $A_{\star,\star} \otimes_{r,H^{\star,\star},l} A_{\star,\star}$ is a free $H^{\star,\star}$-module.

Lemma

There is a $H^{\star,\star}$-bilinear (on the left) perfect pairing between $A_{\star,\star} \otimes_{r,H^{\star,\star},l} A_{\star,\star}$ and $A^{\star,\star} \otimes_{r,H^{\star,\star},l} A^{\star,\star}$:

$$\langle \alpha \otimes \beta, C \otimes D \rangle = \langle \alpha, C \langle \beta, D \rangle \rangle = \langle \lambda (\langle \beta, D \rangle) \cdot \alpha, C \rangle$$

It is well defined and the basis dual to the $P^I \otimes P^J$ is the basis of the $\theta(I)^\star \otimes \theta(J)^\star$.

Definition

We define a comultiplication $\Psi_\star \colon A_{\star,\star} \to A_{\star,\star} \otimes_{\mathrm{r},H^{\star,\star},\mathrm{l}} A_{\star,\star}$ so that for all $\alpha \in A_{\star,\star}$ and $C \otimes D \in A^{\star,\star} \otimes_{\mathrm{r},H^{\star,\star},\mathrm{l}} A^{\star,\star}$, we have :

$$\langle \Psi_\star \alpha, C \otimes D \rangle = \langle \alpha, CD \rangle$$

One can check that $\Psi_\star$ is a ring morphism and that it is $H^{\star,\star}$-linear.

Proposition

$$\Psi_\star(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i \qquad \Psi_\star(\tau_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \tau_i + \tau_k \otimes 1$$

Proposition

$$\Psi_\star(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i \qquad \Psi_\star(\tau_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \tau_i + \tau_k \otimes 1$$

For the first identity, we have to show $\langle \xi_k, CD \rangle = \sum_{i=0}^{k} \left\langle \xi_{k-i}^{2^i} \otimes \xi_i, C \otimes D \right\rangle$. One may assume that $\langle \xi_i, D \rangle \in \{0, 1\}$. Then, we have to show:

$$\langle \xi_k, CD \rangle = \sum_{i=0}^{k} \left\langle \xi_{k-i}^{2^i}, C \right\rangle \langle \xi_i, D \rangle$$

Using formulas for $F(v)$ and $F(v^{2^?})$, we compute

$$CD(v) = C(\sum_{i \geq 0} \langle \xi_i, D \rangle \, v^{2^i}) = \sum_i \sum_j \left\langle \xi_j^{2^i}, C \right\rangle \langle \xi_i, D \rangle \, v^{2^{i+j}} = \sum_k \langle \xi_k, CD \rangle \, v^{2^k}$$

The other identity follows from the computation of $CD(u)$.

Definition

We let $I \subset A_{\star,\star}$ be the ideal generated by the ξ_i for $i \geq 1$. We showed that $\Psi_\star(I) \subset A_{\star,\star} \otimes I + I \otimes A_{\star,\star}$. Then, we have an induced comultiplication:

$$\overline{\Psi}_\star \colon A_{\star,\star}/I \to A_{\star,\star}/I \otimes_{\mathrm{d},H^{\star,\star},\mathrm{g}} A_{\star,\star}/I$$

We let $B^{\star,\star} \subset A^{\star,\star}$ the orthogonal $I^\perp$ of $I \subset A_{\star,\star}$. If follows that $B^{\star,\star}$ is a subring of $A^{\star,\star}$ (that contains $H^{\star,\star}$).

If $C, D \in B^{\star,\star}$ and $\alpha \in I$, $\langle \alpha, CD \rangle = \langle \Psi_\star(\alpha), C \otimes D \rangle = 0$, and $CD \in B^{\star,\star}$.

Definition

For $i \geq 0$, we let $Q_i \in A^{2^{i+1}-1, 2^i-1}$ be the element dual to τ_i from the basis of $A_{\star,\star}$ consisting of monomials $\omega(I)$. We have $Q_i \in B^{\star,\star}$.

Q_i is also the dual of the class of $\tau_i \in A_{\star,\star}/I$ in the basis consisting of monomials involving the τ_j (of degree at most 1 in each variable).

Definition

More generally, for any finite subset I of $\mathbf{N}$, we define $Q_I \in B^{\star,\star}$ as the dual of $\tau_I = \prod_{i \in I} \tau_i$ in the basis of such monomials.

Proposition

If I and J are two finite subsets of $\mathbf{N}$, then $Q_I Q_J$ is:

- *$Q_{I \sqcup J}$ is I and J are disjoint.*
- *0 otherwise.*

We know that $\overline{\Psi}_\star \tau_i = 1 \otimes \tau_i + \tau_i \otimes 1$, then $\overline{\Psi}_\star \tau_K = \sum_{I' \sqcup J' = K} \tau_{I'} \otimes \tau_{J'}$.

Then, we use:

$$Q_I Q_J = \sum_K \left\langle \overline{\Psi}_\star \tau_K, Q_I \otimes Q_J \right\rangle Q_K$$

Corollary

- $Q_i Q_i = 0$
- $Q_i Q_j = Q_j Q_i$.
- $Q_I = \prod_{i \in I} Q_i$.

Definition (Margolis homology)

For any $\mathcal{X} \in \mathcal{H}_\bullet(k)$, we denote $\widetilde{MH}_i^{p,q}(\mathscr{X})$ the homology at $\widetilde{H}^{p,q}(\mathcal{X})$ of the complex:

$$\dots \xrightarrow{Q_i} \widetilde{H}^{p-2^{i+1}+1,q-2^{i+1}+1}(\mathcal{X}) \xrightarrow{Q_i} \widetilde{H}^{p,q}(\mathcal{X}) \xrightarrow{Q_i} \widetilde{H}^{p+2^{i+1}-1,q+2^i-1}(\mathcal{X}) \xrightarrow{Q_i} \dots$$

Proposition

$Q_0 = \beta$.

For degree reasons, $Q_0 = x\beta$ for $x \in \mathbf{Z}/2\mathbf{Z}$. We know $Q_0 \neq 0$. Then, $x = 1$.

For $n \geq 0$, we introduce its digits in base 2: $n = \sum_{i\geq 0} \varepsilon_i 2^i$. We set $\sigma(n) = \sum_i \varepsilon_i$.

Then, I set (personal notation) $Q(n) = \prod_i Q_i^{\varepsilon_i}$. For instance, $Q_i = Q(2^i)$.
(Similarly, $\tau(n) = \prod_i \tau_i^{\varepsilon_i}$.)

Proposition

For any $i \geq 0$, $\Psi^\star(Q_i) \in B^{\star,\star} \otimes_{H^{\star,\star}} B^{\star,\star}$. More precisely,

$$\begin{aligned}\Psi^\star(Q_i) &= \sum_{n+n'=2^i} \rho^{\sigma(n)+\sigma(n')-1} Q(n) \otimes Q(n') \\ &= 1 \otimes Q_i + Q_i \otimes 1 + \sum_{\substack{n+n'=2^i \\ n,n'\geq 1}} \rho^{i-v_2(n)} Q(n) \otimes Q(n')\end{aligned}$$

Lemma

For all $n, n' \geq 0$, we have $\tau(n)\tau(n') = \rho^s \tau(n+n')$ in $A_{\star,\star}/I$ where s is the number of carries when computing $n + n'$ in base 2 (this number is $\sigma(n) + \sigma(n') - \sigma(n+n')$).

Follows from $\tau_i^2 = \rho\tau_{i+1}$.

For the proof of the proposition, we introduce:

Definition (Milnor basis)

We identify sequences $I = (\varepsilon_0, r_1, \varepsilon_1, \dots)$ as before and tuples $(\varepsilon_\bullet, r_\bullet)$. To these are attached elements $\omega(I) = \tau_\bullet^{\varepsilon_\bullet}\xi_\bullet^{r_\bullet}$ which constitute a basis of $A_{\star,\star}$ as a $H^{\star,\star}$-module. We denote $\rho(\varepsilon_\bullet, r_\bullet) \in A^{\star,\star}$ the elements of the dual basis. Note that $\rho(\varepsilon_\bullet, 0) = Q_{\{i,\varepsilon_i \neq 0\}} = \prod_i Q_i^{\varepsilon_i} \in B^{\star,\star}$. We also define $\mathscr{P}^{r_\bullet} = \rho(0, r_\bullet)$.

One can write $\Psi^\star(Q_i) = \sum_{\substack{(\varepsilon_\bullet, r_\bullet) \\ (\varepsilon'_\bullet, r'_\bullet)}} c_{(\varepsilon_\bullet, r_\bullet),(\varepsilon'_\bullet, r'_\bullet)} \rho(\varepsilon_\bullet, r_\bullet) \otimes \rho(\varepsilon'_\bullet, r'_\bullet)$ with

$$c_{(\varepsilon_\bullet, r_\bullet),(\varepsilon_\bullet, r_\bullet)} = \left\langle \tau_\bullet^{\varepsilon_\bullet}\xi_\bullet^{r_\bullet} \otimes \tau_\bullet^{\varepsilon'_\bullet}\xi_\bullet^{r'_\bullet}, \Psi^\star(Q_i) \right\rangle = \left\langle \tau_\bullet^{\varepsilon_\bullet}\tau_\bullet^{\varepsilon'_\bullet}\xi_\bullet^{r_\bullet + r'_\bullet}, Q_i \right\rangle$$

Q_i is orthogonal to the ideal generated by ξ_i for $i \geq 0$. Then, the nonzero coefficients may appear only for $r_\bullet = r'_\bullet = 0$. Denote $n = \sum_i \varepsilon_i 2^i$ and $n' = \sum_i \varepsilon'_i 2^i$, we have:

$$\langle \tau(n)\tau(n'), Q_i \rangle = \rho^{\sigma(n)+\sigma(n')-1} \left\langle \tau(n+n'), Q(2^i) \right\rangle = 0 \text{ unless } n + n' = 2^i$$

We showed that:

$$\Psi^\star(Q_i) = \sum_{n+n'=2^i} \rho^{\sigma(n)+\sigma(n')-1} Q(n) \otimes Q(n')$$

which implies:

$$\Psi^\star(Q_i) = 1 \otimes Q_i + Q_i \otimes 1 + \sum_{\substack{n+n'=2^i \\ n,n' \geq 1}} \rho^{i-v_2(n)} Q(n) \otimes Q(n')$$

It gives formulas for the computation of $Q_i(xy)$ in terms of images of x and y by compositions of some Q_j (for $j < i$).

Proposition

$$\rho(\varepsilon_\bullet, r_\bullet) = Q_{\{i,\varepsilon_i \neq 0\}} \mathscr{P}^{r_\bullet}$$

(where $\mathscr{P}^{r_\bullet} = \rho(0, r_\bullet)$)

This means $\rho(\varepsilon_\bullet, r_\bullet) = \rho(\varepsilon_\bullet, 0)\rho(0, r_\bullet)$.

Proposition

For any $n \geq 1$, we denote $q_n \in A^{\star,\star}$ the element in the Milnor basis $\rho(-,-)$ that is dual to $\xi_n \in A_{\star,\star}$. Then, $Q_n = [\beta, q_n] = \beta q_n + q_n \beta$.

We have to show $q_n\beta = Q_n + \beta q_n$. Q_n and βq_n belong to the Milnor basis (they are the duals of τ_n and $\tau_0\xi_n$). We consider pairings

$$\langle \omega(I), q_n\beta \rangle = \langle \Psi_\star(\omega(I)), q_n \otimes \beta \rangle$$

Let $J \subset A_{\star,\star}$ the ideal generated by τ_k, $k \geq 1$ and ξ_k, $k \geq 1$. (Then $A_{\star,\star}/J = H^{\star,\star}[\tau_0]/(\tau_0^2)$.) As $\langle J, \beta \rangle = 0$, it suffices to examine $\Psi_\star(\omega(I))$ in the quotient $A_{\star,\star} \otimes_{\mathrm{r}, H^{\star,\star}, \mathrm{l}} A_{\star,\star}/J$. There we have:

$$\overline{\Psi}_\star(\xi_k) = \xi_k \otimes 1 \qquad \overline{\Psi}_\star(\tau_k) = \xi_k \otimes \tau_0 + \tau_k \otimes 1$$

Then, the only $\omega(I)$ such that $\overline{\Psi}_\star(\omega(I))$ contains a term $\xi_n \otimes \tau_0$ are $\tau_0\xi_n$ and τ_n and then the coefficient is 1.

Proposition

For any $n \geq 0$, $P^n = \mathscr{P}^{(n,0,0,\dots)}$.

This means that in the Milnor basis, P^n is dual to ξ_1^n.
We already know that $\langle \omega(J), P^n \rangle = 0$ if $(n, 0, \dots) < J$. It remains only the cases $J = (k, 0, \dots)$ with $k < n$. But then,

$$\left\langle \xi_1^k, P^n \right\rangle \in H^{2(n-k),n-k} = 0 \text{ unless } k = n$$

We want to understand to some extend the action of the Steenrod algebra on Thom classes of vector bundles.
Some remarks:

- An operation $\mathscr{P}^{r_\bullet}$ (dual in the Milnor basis of some monomial involving the ξ_i) is in $A^{2n,n}$ for some n.
- The operation Q_i is in $A^{p,q}$ for $p > 2q$.
- $\rho(\varepsilon_\bullet, r_\bullet) = Q_{\{i,\varepsilon_i \neq 0\}} \mathscr{P}^{r_\bullet}$

Proposition

The operations Q_i and more generally the operations $\rho(\varepsilon_\bullet, r_\bullet)$ for $\varepsilon_\bullet \neq 0$ vanish on $H^{2\star,\star}(X) = CH^\star(X)/2$ and on $\widetilde{H}^{2\star,\star}(\mathrm{Th}_X V)$ (with V a vector bundle of rank r on $X \in Sm/k$).

In particular, such operations kill the Thom class $t_V \in \widetilde{H}^{2r,r}(\mathrm{Th}_X V)$ of any vector bundle.

Now, we focus on the action of operations $\mathscr{P}^{r_\bullet}$ on Thom classes t_V and we shall start with the case of line bundles.

Proposition

Let $X \in Sm/k$. If L is a line bundle on X. Then, $\lambda(c_1(L)) = \sum_{i\geq 0} \xi_i \otimes c_1(L)^{2^i}$.

We already did this computation in the universal case of $v = c_1(\mathscr{O}(1))$ on $\mathbf{P}^\infty$.

Corollary

Let $X \in Sm/k$. If L is a line bundle on X. We let $t_L \in \widetilde{H}^{2,1}(\mathrm{Th}_X\, L)$ be the Thom class. Then,

$$\lambda(t_L) = \sum_{i\geq 0} \xi_i \otimes \left(c_1(L)^{2^i-1} t_L\right) \in A_{\star,\star} \otimes_{H^{\star,\star}} \widetilde{H}^{\star,\star}(\mathrm{Th}_X\, L)$$

We can do the computation in $\mathbf{P}(L \oplus \mathscr{O}_X)$ where $t_L = \xi + c_1(L)$ with $\xi = c_1(\mathscr{O}(1))$. It suffices to show:

$$\xi^{2^i} + c_1(L)^{2^i} = c_1(L)^{2^i-1}(\xi + c_1(L))$$

i.e., $\xi^{2^i} = c_1(L)^{2^i-1}\xi$, which follows from the identity $\xi^2 + c_1(L)\xi = 0$ (definition of Chern classes of the bundle $L \oplus \mathscr{O}$).

Proposition

Let $r_\bullet = (r_1, r_2, \dots)$ a sequence of integers as above. We have a monomial $\xi^{r_\bullet}$. Let $d \geq 0$. We denote $P \in \mathbf{F}_2[x_1, \dots, x_d]$ the symmetric polynomial

$$P = \sum_{\substack{(j_1, \dots, j_d) \in \mathbf{N}^d \\ \xi_{j_1} \cdots \xi_{j_d} = \xi^{r_\bullet}}} \prod_{i=1}^{d} x_i^{2^{j_i} - 1}$$

We denote $R \in \mathbf{F}_2[c_1, \dots, c_d]$ the unique polynomial such that if we substitute to c_i the ith elementary symmetric function of the x_i we get P. Then, for any vector bundle V of rank d on $X \in Sm/k$, we have:

$$\mathscr{P}^{r_\bullet}(t_V) = R(c_1(V), \dots, c_d(V)) \cdot t_V$$

(Note that the formula will stabilise for big enough d, for example $d \geq \sum_i (2^i - 1) r_i$.) As we did before, using the splitting principle, one may assume that $V = L_1 \oplus \cdots \oplus L_d$ for line bundles L_i.

$V = L_1 \oplus \cdots \oplus L_d$. We set $x_i = c_1(L_i)$. We have to show:

$$\mathscr{P}^{r_\bullet}(t_V) = \left(\sum_{\substack{(j_1,\ldots,j_d)\in\mathbf{N}^d \\ \xi_{j_1}\cdots\xi_{j_d}=\xi^{r_\bullet}_\bullet}} \prod_{i=1}^{d} x_i^{2^{j_i}-1} \right) \cdot t_V$$

From the computation of $\lambda(t_{L_i})$, we get:

$$\lambda(t_V) = (\prod_{i=1}^{d} \sum_{j=0}^{\infty} \xi_j \otimes x_i^{2^j-1}) \cdot t_V$$

The class $\mathscr{P}^{r_\bullet}(t_V)$ is the coefficient of the monomial $\xi^{r_\bullet}_\bullet$ in this expansion, which gives the expected result.

Here is general formula again: $P = \sum_{\substack{(j_1,\dots,j_d)\in\mathbf{N}^d \\ \xi_{j_1}\cdots\xi_{j_d}=\xi^r_\bullet}} \prod_{i=1}^d x_i^{2^{j_i}-1}$.

Corollary

$P^n(t_V) = C_n(V)\cdot t_V$ where $C_n(V) = C_n(c_1(V),\dots,c_d(V))$ is the polynomial in the symmetric functions corresponding to $\displaystyle\sum_{\substack{I\subset\{1,\dots,d\} \\ \#I=n}} \prod_{i\in I} x_i$.

Corollary

Remember q_n is the operation dual to ξ_n. Then, $q_n(t_V) = s_{2^n-1}(V)\cdot t_V$ where $s_j\colon K_0(X)\to\oplus_i H^{2i,i}(X)$ is the additive natural transformation such that $s_j(c_1(L)) = c_1(L)^j$ for line bundles L.

Here, we have $P = \sum_{i=1}^d x_i^{2^j-1}$.