# *Summer school*
# Motives and Milnor conjecture

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Contents



## 1. Aim of the summer school

As the first objective of the summer school, we want to offer the audience an introduction to the homotopy theory of schemes and to the theory of mixed motives.

The second objective is to provide a detailed overview of the proof of the Milnor conjecture, as an illustration of the theories introduced in the first part.

## 2. Abstract of courses

A. **Chow motives and motives of quadrics.** (3 talks)

The purpose of this course is to present the construction of the *Rost motive* attached to a norm quadric and to establish some of its properties inside the category of Chow motives (as stated in [Voe03a, th. 4.4]). As a preliminary talk, the category of Chow motives as well as the category of Grothendieck pure motives will be constructed.

A.1. *Chow motives.* Definition, basic properties.

A.2. *Motives of quadrics I.*

A.3. *Motives of quadrics II.*

*Pre-requisite.* Basic algebraic geometry ([Har77] or [DLØ+07, Levine]) and intersection theory ([Ful98]).

B. **Motivic complexes and motivic cohomology.** (5 talks)

The theory of motivic complexes has been conjectured by Beilinson and Lichtenbaum. It is conceived as an extension of the theory of pure motives over a field $k$ but a distinctive feature is that any algebraic $k$-scheme gives rise to a motivic complex. The latter is a particular exemple of a *mixed motive* by opposition to *pure motives*. The terminology is directly taken from Deligne theory of mixed Hodge structure.

The course will be focused on the definition of motivic complexes of Voevodsky. It is based on the consideration of special correspondences, called *finite correspondences*, which are not considered up to an equivalence relation. This is possible with the help of Serre intersection theory. These correspondences give rise to the notion of a sheaf with transfers which is central in the theory: motivic complexes will be particular kind of complexes of sheaves with transfers.

The next particularity of the theory of Voevodsky is its use of the $\mathbb{A}^1$-homotopy equivalence and the related property of homotopy invariance of cohomology (with respect to $\mathbb{A}^1$). The study of homotopy invariant sheaves with transfers is the technical heart of the theory. It will be exposed through the first three courses.

Then the theory of motivic complexes can go on, especially with the definition of the *Tate motivic complex* $\mathbb{Z}(n)$ and the computation of its cohomology in the Nisnevich or étale topology.

B.1. *Sheaves with transfers I.* Finite correspondences. Definition, first properties (over a smooth $k$-schemes, $k$ a field).

B.2. *Sheaves with transfers II.* Homotopy invariance, first results on cohomology.

B.3. *Sheaves with transfers III.* Main theorem on cohomology: homotopy invariance, Gersten resolution.

B.4. *Motivic cohomology I.* For 2 talks:

Motivic complexes. Definition of the complex $\mathbb{Z}(n)$, computation of $\mathbb{Z}(1)$. Motivic cohomology and Milnor K-theory, Chow group. Duality and the embedding of Chow motives.

Lichtenbaum motivic cohomology. Case of rational coefficients. Computation of $\mathbb{Z}/l(n)$ in the étale topology.

B.5. *Motivic cohomology II.*

*References.* [VSF00, chap. 3 and 5], [MVW06]

*Pre-requisite.* Basic algebraic geometry ([Har77] or [DLØ+07, Levine]), intersection theory ([Ser65]), sheaf theory ([Mil80] or [DLØ+07, Levine]).

C. **Homotopy theory of schemes.** (5 talks)

One of the roots of the theory of Voevodsky is that one should be able to incorporate ideas from modern algebraic topology into algebraic geometry, and especially in the ongoing theory of motives.[1] This program has been realized by Morel and Voevodsky as the *(stable) homotopy theory of schemes* which will be presented in this course.

[1]Recall the first idea of proof of the Milnor conjecture from Voevodsky used the existence of analogs of Morava K-theory in algebraic geometry.

The theory will be essentially used in the proof of the Milnor conjecture as the natural framework in which constructing cohomological operations such as Steenrod operations on motivic cohomology (course D).

C.1. *The unstable homotopy category I.* For 2 talks: Recall on model categories. The unstable homotopy category (any base).

C.2. *The unstable homotopy category II.*

C.3. *The unstable homotopy category III.* Purity, the degree map ([Voe03a, sec. 2, th. 2.11]).

C.4. *The stable homotopy category I.* For 2 talks: Spectra, symmetric spectra. Stable model category. Exemple of spectra: Motivic Eilenberg-Mac Lane, algebraic cobordism.

C.5. *The stable homotopy category II.*

*References.* [MV99], [DLØ+07], [Jar00].

*Pre-requisite.* The reading of the first chapters of [DLØ+07] is recommended. Simplicial objects, basic knowledge of model categories ([Qui67], [Hov99]). Basic algebraic geometry ([Har77]) and sheaf theory ([Mil80]).

D. **The Steenrod operations.** (4 talks)

The Steenrod operations on torsion singular cohomology is a fundamental piece of algebraic topology and was intensively studied for more than fifty years. In his second proof of the Milnor conjecture, Voevodsky introduced the analog of these operations in mod 2 motivic cohomology as the final tool which allows to finish the induction step (see the abstract of course F). The construction of these operations is a difficult part of the theory which took a long time to emerged. This is a whole paper by Voevodsky: [Voe03b].

The four talks of this course will be devoted to the exposition of that paper, using both courses B and C.

D.1. *The steenrod operations I.*

D.2. *The steenrod operations II.*

D.3. *The steenrod operations III.*

D.4. *The steenrod operations IV.*

*References.* [Voe03b]

E. **Galois cohomology and Milnor K-theory.** (2 talks)

The aim of this elementary course is to introduce Milnor K-theory and Galois cohomology with their basic properties in order to formulate the Milnor conjecture. The original point of view of Milnor, using the Witt ring of quadratic forms will also be presented.

E.1. *Milnor K-theory, Witt ring.*

E.2. *Galois cohomology, Hilbert 90, Milnor conjecture.*

*Pre-requisite.* Group cohomology [Ser94] (chap. 1, primarily §2), Grothendieck ring of monoidal abelian categories.

F. **Proof of the Milnor conjecture.** (6 talks)

The aim of this serie of talks is to explain the proof of the Milnor conjecture for a field $k$, [Voe03a, th. 7.4].

The proof is based on a reformulation of the conjecture as the vanishing of the Lichtenbaum motivic cohomology group of $k$

$$\mathrm{H}_{\mathrm{L}}^{w+1,w}(k,\mathbb{Z}_{(2)})$$

for all $w \geq 0$ – this vanishing property is called $\mathrm{H90}(w,2)$ in [Voe03a, Def. 6.4]. The fact this property is equivalent to the Milnor conjecture is a key ingredient: see [Voe03a, sec. 6].[2]

The proof goes on by induction on $w$, the first cases being well known (it will be explained in Course E.2). The inductive step can be divided into the following sequence of arguments:

- Using computations with Galois cohomology (see [Voe03a, sec. 5]), one restricts to prove the existence, for each *symbol* $\underline{a}$ in the Milnor K-group $K_w^M(k)$, of a function fields $K_{\underline{a}}$ satisfying the following properties:
  (i) $\underline{a}$ is 2-divisible in $\mathrm{K}_w^M(\mathrm{K}_{\underline{a}})$.
  (ii) the induced morphism
  $$\mathrm{H}_{\mathrm{L}}^{w+1,w}(k,\mathbb{Z}_{(2)}) \to \mathrm{H}_{\mathrm{L}}^{w+1,w}(K_{\underline{a}},\mathbb{Z}_{(2)})$$
  is injective.

  The field $K_{\underline{a}}$ is in fact the function field of certain quadrics called *norm quadrics* (introduced in Course A).[3]
- Property (i) follows relatively easily ([Voe03a, Prop. 4.1]).
- All the problem is to get property (ii). The main ingredients to get it are:
  - The existence of the Rost motive ([Voe03a, Th. 4.4], explained in Course A).
  - Steenrod operations on motivic cohomology mod 2 and the associated motivic Margolis homology. (explained in Course D)

F.1. *Proof I.*

F.2. *Proof II.*

F.3. *Proof III.*

F.4. *Proof IV.*

F.5. *Proof V.*

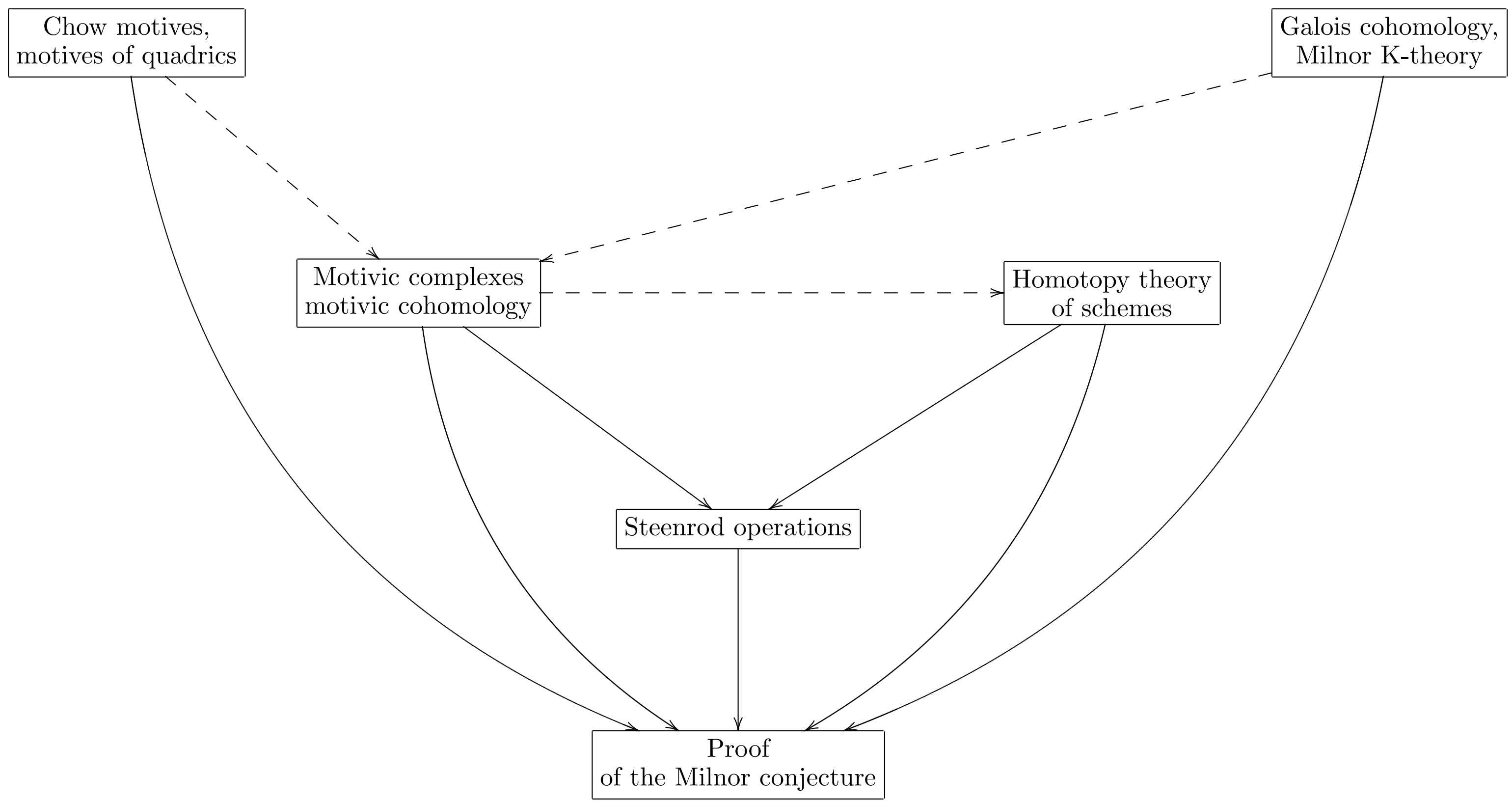
F.6. *Proof VI.*

*References.* [Voe03a]

[2] The original idea is due to Suslin and Voevodsky [SV00] and has been nicely extended in [GL01].

[3] They are also sometimes referred to as *splitting varieties*, and their function field as a *splitting field*.

## 3. Leitfaden


Chow motives,
motives of quadrics
Galois cohomology,
Milnor K-theory
Motivic complexes
motivic cohomology
Homotopy theory
of schemes
Steenrod operations
Proof
of the Milnor conjecture

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